Gravitational Wave Simulations

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I. Schwarzschild Geometry

The Schwarzschild metric is used to characterize the geometry of spacetime outside a spherically symmetric mass at rest. As derived in the section IV of ”Gravitational Wave Theories”, the line element ds^2 of the Schwarzschild geometry can be represented as

$$(ds)^2 = -(1-2M/r)(dt)^2 + \frac{1}{1-2M/r}(dr)^2 + r^2[(d\theta^2) + \sin^2\theta(d\phi)^2] \quad (1)$$

Thus, the Schwarzschild metric is represented as

$$g_{\alpha\beta} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \quad (2)$$

The geodesic equation in the four-dimensional spacetime is represented as

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \quad (3)$$

where λ is called affine parameter, $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\ell}\left(\frac{\partial g_{\ell\gamma}}{\partial x^\beta} + \frac{\partial g_{\ell\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\ell}\right) \quad (4)$$

Consider motions on an equatorial plane, $\theta = \pi/2$, the Schwarzschild line element in (1) is reduced to

$$(ds)^2 = -(1-2M/r)(dt)^2 + \frac{1}{1-2M/r}(dr)^2 + r^2(d\phi)^2 \quad (5)$$

The geodesic equation in (3) of a free test particle in a Schwarzschild spacetime can thus be expressed as

$$\frac{\partial^2 t}{\partial\lambda^2} = \frac{2M}{2Mr-r^2}\frac{\partial t}{\partial\lambda}\frac{\partial r}{\partial\lambda} \quad (6)$$

$$\frac{\partial^2 r}{\partial\lambda^2} = -\frac{M}{2Mr-r^2}\left(\frac{\partial r}{\partial\lambda}\right)^2 + \frac{M(2M-r)}{r^3}\left(\frac{\partial t}{\partial\lambda}\right)^2 - (2M-r)\left(\frac{\partial\phi}{\partial\lambda}\right)^2 \quad (7)$$

$$\frac{\partial^2\theta}{\partial\lambda^2} = 0 \quad (8)$$

$$\frac{\partial^2\phi}{\partial\lambda^2} = \frac{2}{r}\frac{\partial r}{\partial\lambda}\frac{\partial\phi}{\partial\lambda} \quad (9)$$

Eqn.(9) is rearranged as

$$\frac{\partial^2\phi/\partial\lambda^2}{\partial\phi/\partial\lambda} = -\frac{2\partial r/\partial\lambda}{r}$$

implying that

$$\ln\frac{\partial\phi}{\partial\lambda} = -2\ln r + \ln c_1$$

or

$$\frac{\partial\phi}{\partial\lambda} = \frac{c_1}{r^2} \quad (10)$$

where c_1 is a constant of integration.

Similarly, (6) is rearranged as

$$\frac{\partial^2 t/\partial\lambda^2}{\partial t/\partial\lambda} = \frac{2M\partial r/\partial\lambda}{r(2M-r)}$$

implying that

$$\ln\frac{\partial t}{\partial\lambda} = \ln\left(\frac{1}{1-2M/r}\right) + \ln c_2$$

or

$$\frac{\partial t}{\partial\lambda} = \frac{c_2}{1-2M/r} \quad (11)$$

where c_2 is another constant of integration.

A geodesic satisfies the Euler-Lagrange equation

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial x^\mu/\partial\sigma}\right) = \frac{\partial L}{\partial x^\mu} \quad (12)$$

where L is the Lagrangian. If L is explicitly independent of a particular x^μ, then $\partial L/\partial x^\mu = 0$, implying that $\partial L/(\partial x^\mu/\partial\sigma)$ is a conserved quantity with respect to σ, a real argument. For null geodesic, describing massless particles such as photons, σ is replaced with the affine parameter λ.

In a flat spacetime, we have

$$dt^2 = dx^2 - ds^2 \quad (13)$$

The time variable t in the geodesic frame ($dx = 0$) is called the proper time τ, namely,

$$d\tau^2 = -ds^2 \quad (14)$$

which is less than or equal to dt^2 In a general spacetime, (14) can be generalized to

$$d\tau = \sqrt{-ds^2} = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \quad (15)$$

by applying the definition of a line element in spacetime.

By Hamilton's principle, a geodesic must satisfy

$$\int \delta L d\lambda = 0 \quad (16)$$

If

$$L = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \tag{17}$$

then (16) is reduced to

$$\delta \int d\tau = 0 \tag{18}$$

which implies that the proper time along the geodesic is the minimum among all possible paths, and (17) is the Lagrangian of the system.

By substituting (17), with ϕ-independence, into (12), we derive

$$\begin{aligned}\frac{\partial L}{\partial \phi/\partial\lambda} &= \frac{-2g_{44}(d\phi/d\lambda)}{2\sqrt{-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)}} \\ &= -g_{44}\frac{d\phi}{d\tau} = -c_1 \end{aligned} \tag{19}$$

where c_1 is a constant. If the g_{44} of the Schwarzschild metric in (5) is adopted, (19) is reduced to

$$r^2\frac{d\phi}{d\tau} = c_1 \tag{20}$$

which is the conservation of angular momentum. Note that (10) for light has the same form as (20), with λ changed to τ.

Similarly, by applying $L = \sqrt{-g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)}$ and its t independence to (12), we have

$$\frac{\partial L}{\partial t/\partial\lambda} = -g_{00}\frac{dt}{d\tau} = c_2 \tag{21}$$

where c_2 is a constant. By substituting g_{00} of the Schwarzschild metric in (2) into (21), we obtain

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = c_2 \tag{22}$$

which is the same as (11).

By substituting the relation of $d\tau$ in (15) into (21), we obtain

$$\begin{aligned} c_2 &= \frac{-g_{00}}{\sqrt{-g_{00} - g_{11}(dr/dt)^2 - r^2[(d\theta/dt)^2 + \sin^2\theta(d\phi/dt)^2]}} \\ &= \frac{\sqrt{-g_{00}}}{\sqrt{1 + g_{11}(dr/dt)^2/g_{00} + r^2[(d\theta/dt)^2 + \sin^2\theta(d\phi/dt)^2]/g_{00}}} \end{aligned} \tag{23}$$

In the Newtonian limit (as if $r \to \infty$), $g_{00} \to -1$ and $g_{11} \to 1$, thus (23) can be simplified as

$$c_2 \simeq \frac{\sqrt{-g_{00}}}{\sqrt{1 - (d\tilde{r}/dt)^2}} \simeq \sqrt{-g_{00}}\left[1 + \frac{1}{2}\left(\frac{d\tilde{r}}{dt}\right)^2\right] \tag{24}$$

where $d\tilde{r} = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2}$ is the spatial displacment in spherical coordinates. If we substitute the g_{00} of the Schwarzschild metric into (24), we have

$$c_2 = (1 - M/r)[1 + (d\tilde{r}/2dt)^2] \simeq 1 - M/r + \frac{1}{2}\left(\frac{d\tilde{r}}{dt}\right)^2$$

which can be identified as the conservation of energy in Newtonian physics. Therefore, c_2 implies the quantity of energy conservation.

A light ray is parameterized by λ as $x^\alpha(\lambda)$, which satisfies

$$ds^2 = 0$$

implying that $(ds/d\lambda)^2 = 0$, or explicitly

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{25}$$

By substituting (10), (11) and the Schwarzchild metric in (5) into (25), we have

$$-\frac{c_2^2}{1 - 2M/r} + \frac{(\partial r/\partial\lambda)^2}{1 - 2M/r} + \frac{c_1^2}{r^2} = 0$$

leading to

$$\left(\frac{\partial r}{\partial \lambda}\right)^2 = c_2^2 - \frac{c_1^2(r - 2M)}{r^3} \tag{26}$$

which can be reduced to

$$\frac{1}{c_1^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - W_{\text{eff}}(r) \tag{27}$$

where

$$\begin{aligned} b &= \frac{c_1}{c_2} \\ W_{\text{eff}}(r) &= \frac{1}{r^2}(1 - 2M/r) \end{aligned} \tag{28}$$

Since $(\partial r/\partial\lambda)^2/c_1^2 > 0$, (27) implies that

$$W_{\text{eff}}(r) \le \frac{1}{b^2} \tag{29}$$

which implies a turning point at r that satisfies $W_{\text{eff}}(r) = 1/b^2$.

By substituting $c_1 = r^2(\partial\phi/\partial\lambda)$, derived from (10), into (26), we obtain

$$\begin{aligned} \frac{d\phi}{dr} &= \frac{\partial\phi/\partial\lambda}{\partial r/\partial\lambda} \\ &= \frac{b}{r^2\sqrt{1 - (b^2/r^2)(1 - 2M/r)}} \end{aligned} \tag{30}$$

where ϕ and r can be seen as the angle and distance in polar coordinate, with the mass as the origin. Thus, we can plot out the trajectory of light coming from $r = \infty$ by Eqn.(30).

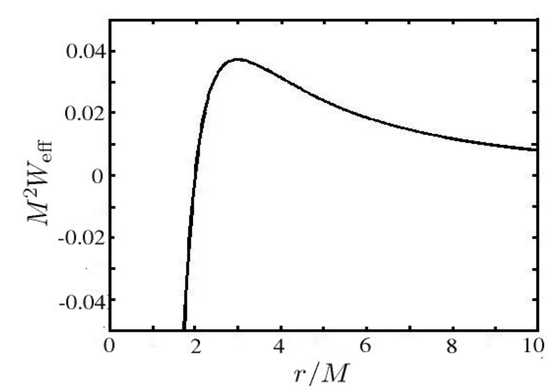

Fig. 1. Distribution of $W_{\text{eff}}(r)$ in (28).

Fig.1 shows the distribution of $W_{\rm eff}(r)$ in (28), where a peak occurs at $r = 3M$. By taking the derivative of (28) with respect to r, we have

$$\frac{dW_{\rm eff}}{dr} = \frac{2}{r^3}\left(\frac{3M}{r} - 1\right)$$

which is equal to zero at the turning point $r = 3M$, at which the maximum value is

$$W_{\rm eff,max} = \frac{1}{27M^2}$$

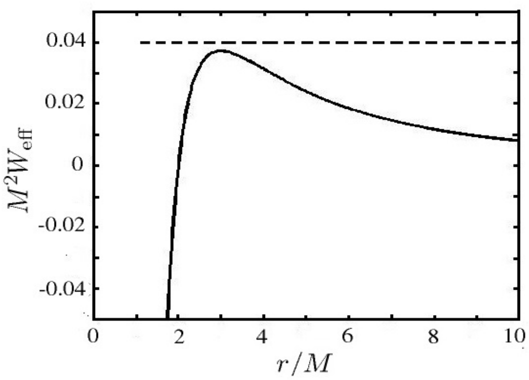

Fig. 2. Distribution of $W_{\rm eff}(r)$ (———), $b = 5M$. The dashed line is $1/b^2$.

Fig.2 shows the distribution of $W_{\rm eff}(r)$, with $b = 5M < 3\sqrt{3}M$. Fig.3 shows a light ray, which takes a plunge orbit. It is obtained by solving (30) with the fourth-order Runge-kutta method. Since $W_{\rm eff} \neq 1/b^2$, there is no turning point.

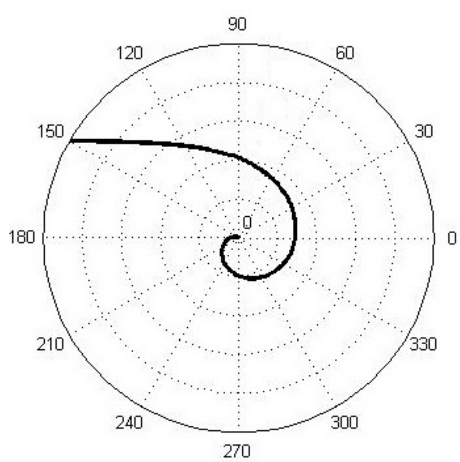

Fig. 3. Light ray takes a plunge orbit when $b = 5M < 3\sqrt{3}M$. The outermost circle is at $r = 100$.

Next, consider the case with $b > 3\sqrt{3}M$, with the distribution of $W_{\rm eff}(r)$ shown in Fig.4. The line of $1/b^2$ intersects with $W_{\rm eff} = 1/b^2$ near $r = 4M$, which is a turning point due to the constraint in (29). The photon is scattered, with its orbit shown in Fig.5.

The case with $b = 3\sqrt{3}M$ leads to a circular orbit, in which light gets trapped. However, the orbit is numerically unstable.

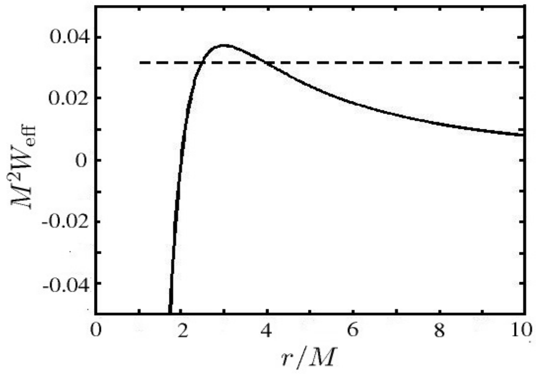

Fig. 4. Distribution of $W_{\rm eff}(r)$ (———), $b = 5.65M$. The dashed line is $1/b^2$.

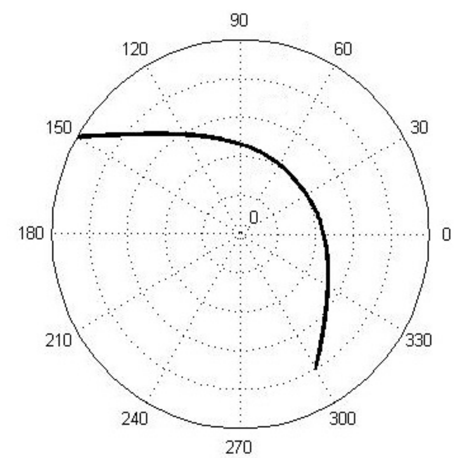

Fig. 5. Light ray takes a scatter orbit when $b = 5.65M > 3\sqrt{3}M$. The outermost circle is at $r = 100$.

II. Kerr Geometry

The angular momentum of a black-hole can be observed from the redshift of its inner accretion disk reflection. In geometrized unit, $c = G = 1$, M is in unit of length, and $a = J/M$ also has a unit of length. The horizon radius of a Kerr black-hole, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$, exists only when $a \le M$. If M is normalized to 1 (unitless), then the normalized a falls within the range from 0 to 1.

The Kerr geometry is used to describe the geometry outside a rotating, spherical mass. In contrast to Schwarzschild black-holes, the properties of a Kerr black-hole are specified in terms of its mass M and angular momentum J. The line element ds^2 is represented as [1]

$$\begin{aligned}(ds)^2 = &-\left(1 - \frac{2Mr}{r^2 + J^2\cos^2\theta/M^2}\right)(dt)^2 \\ &- \frac{4Jr\sin^2\theta}{r^2 + J^2\cos^2\theta/M^2} d\phi dt \\ &+ \frac{r^2 + J^2\cos^2\theta/M^2}{t^2 - 2Mr + J^2/M^2}(dr)^2 \\ &+ (r^2 + J^2\cos^2\theta/M^2)(d\theta^2) \\ &+ \left(r^2 + \frac{J^2}{M^2} + \frac{2J^2 r\sin^2\theta/M}{r^2 + J^2\cos^2\theta/M^2}\right)\sin^2\theta(d\phi)^2\end{aligned} \quad (31)$$

which reduces to the Schwarzchild line element in (1) when $J = 0$.

General geodesics in a Kerr geometry do not lie in a plane, except equatorial orbits. Consider motions on an equatorial plane ($\theta = \pi/2$), the Kerr line element in (31) is reduced to

$$(ds)^2 = -\left(1 - \frac{2M}{r}\right)(dt)^2 - \frac{4aM}{r} d\phi dt + \frac{r^2}{\Delta^2}(dr)^2 + (r^2 + a^2 + 2a^2M/r)(d\phi)^2 \quad (32)$$

where

$$a = J/M$$
$$\Delta = r^2 - 2Mr + a^2$$

Thus, the corresponding Kerr metric is

$$g_{\alpha\beta} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & -2aM/r \\ 0 & r^2/\Delta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2aM/r & 0 & 0 & r^2 + a^2 + 2a^2M/r \end{bmatrix} \quad (33)$$

As in a Schwarzschild geometry, the Euler-Lagrange equation in (12) can be applied to derive the equations of energy conservation and angular-momentum conservation. Consider a light ray, the proper time τ should be replaced with the affine parameter λ. By applying the Kerr metric in (33) with t-independence in (21), we have

$$-c_2 = g_{0\beta}\frac{dx^\beta}{d\lambda} = g_{00}\frac{dt}{d\lambda} + g_{04}\frac{d\phi}{d\lambda}$$

or

$$-c_2 = -\left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda} - \frac{2aM}{r}\frac{d\phi}{d\lambda} \quad (34)$$

Similarly, by applying the Kerr metric in (33) with ϕ-independence in (19), we have

$$c_1 = g_{4\beta}\frac{dx^\beta}{d\lambda} = g_{04}\frac{dt}{d\lambda} + g_{44}\frac{d\phi}{d\lambda}$$

or

$$c_1 = -\frac{2aM}{r}\frac{dt}{d\lambda} + (r^2 + a^2 + 2a^2M/r)\frac{d\phi}{d\lambda} \quad (35)$$

The two conservation equations in (34) and (35) are represented as

$$\begin{bmatrix} -(1-2M/r) & -2aM/r \\ -2aM/r & r^2 + a^2 + 2a^2M/r \end{bmatrix} \begin{bmatrix} dt/d\lambda \\ d\phi/d\lambda \end{bmatrix} = \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} \quad (36)$$

which can be solved to have

$$\begin{bmatrix} dt/d\lambda \\ d\phi/d\lambda \end{bmatrix} = \frac{-1}{\Delta} \begin{bmatrix} r^2 + a^2 + 2a^2M/r & 2aM/r \\ 2aM/r & -(1-2M/r) \end{bmatrix} \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix}$$

or

$$\frac{dt}{d\lambda} = \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2a^2M}{r}\right)c_2 - \frac{2aM}{r}c_1\right] \quad (37)$$

$$\frac{d\phi}{d\lambda} = \frac{1}{\Delta}\left[\frac{2aM}{r}c_2 + \left(1 - \frac{2M}{r}\right)c_1\right] \quad (38)$$

where Δ is the determinant of the 2×2 matrix in (36).

By substituting the Kerr metric into the governing equation of light in (25), we have

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \frac{4aM}{r}\frac{d\phi}{d\lambda}\frac{dt}{d\lambda} + \frac{r^2}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 + \left(r^2 + a^2 + \frac{2a^2M}{r}\right)\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \quad (39)$$

By subtituting $dt/d\lambda$ and $d\phi/d\lambda$ in (37) and (38), respectively, into (39), we obtain

$$\frac{1}{c_1^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - W_{\text{eff}}(r) \quad (40)$$

which has the same form as (27), but with

$$W_{\text{eff}}(r) = \frac{1}{r^2}\left[1 - (a/b)^2 - \frac{2M}{r}(1 - \sigma a/b)^2\right] \quad (41)$$

where $b = c_1/c_2$ and

$$\sigma = \begin{cases} 1, & \text{prograde (corotating) orbit} \\ -1, & \text{retrograde (counterrotating) orbit} \end{cases}$$

Note that by taking $a = 0$, (41) reduces to the case of Schwarzschild geometry in (28).

By using the expressions of $d\phi/d\lambda$ and $dr/d\lambda$ in (38) and (40), respectively, we obtain an expression of $dr/d\phi$ as

$$\frac{dr}{d\phi} = \frac{dr/d\lambda}{d\phi/d\lambda} \quad (42)$$

which describes the light trajectory in terms of a, b, r and M. Note that ϕ and r are the angle and distance in the polar coordinate with the mass at the origin. Thus, light trajectories around a Kerr black hole can be obtained by solving (42) with the fourth-order Runge-Kutta method.

Fig.6 shows a case with $b = 0.1M$, in which both the prograde and retrograde photons follow plunge orbits.

Fig.7 shows a case with $b = 0.125M$, in which both the prograde and retrograde photons follow scatter orbits.

Fig.8 shows a case of scatter orbits which are very close to circular orbits since it is difficult to simulate stable circular orbits by using numerical method.

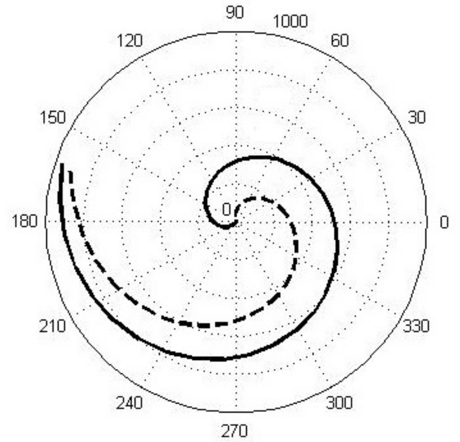

Fig. 6. Prograde orbit (———) and retrograde orbit (− − −), $b = 0.1M$.

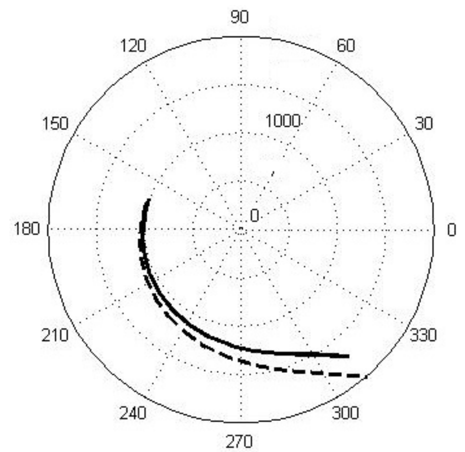

Fig. 7. Prograde orbit (———) and retrograde orbit (− − −), $b = 0.125M$.

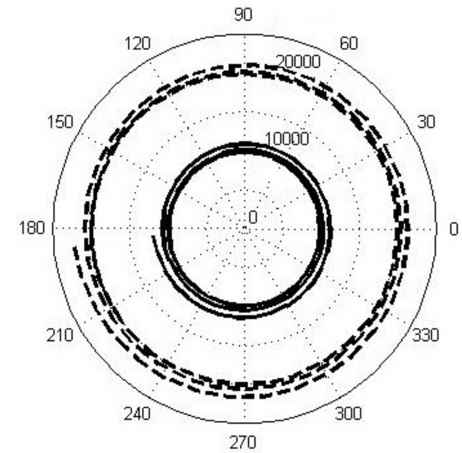

Fig. 8. Prograde orbit (———) and retrograde orbit (− − −), $b = 0.106M$.

III. Gravitational Wave of a Binary System

In linearized gravity, we assume

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{43}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ is a small perturbation. Under the gauge condition of

$$\frac{\partial}{\partial x^\lambda} h'^{\lambda}_{\nu} = 0 \tag{44}$$

the Einstein's equations can be derived as [6]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h'_{\mu\nu}(t, x^\alpha) = -KT_{\mu\nu}(t, x^\alpha) \tag{45}$$

where $K = 16\pi G/c^4$, and

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h\eta_{\mu\nu} \tag{46}$$

By representing $T_{\mu\nu}$ and $h'_{\mu\nu}$ in the frequency domain, we have

$$\begin{aligned} T_{\mu\nu}(t, x^\alpha) &= \int_{-\infty}^{\infty} T_{\mu\nu}(\omega, x^\alpha) e^{-i\omega t} d\omega \\ h'_{\mu\nu}(t, x^\alpha) &= \int_{-\infty}^{\infty} h'_{\mu\nu}(\omega, x^\alpha) e^{-i\omega t} d\omega \end{aligned} \tag{47}$$

which are then substituted into (45) to have

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h'_{\mu\nu}(\omega, x^\alpha) e^{-i\omega t} d\omega \\ &= \int_{-\infty}^{\infty} \left(\nabla^2 + \frac{\omega^2}{c^2}\right) h'_{\mu\nu}(\omega, x^\alpha) e^{-i\omega t} d\omega \\ &= -K \int_{-\infty}^{\infty} T_{\mu\nu}(\omega, x^\alpha) d\omega \end{aligned} \tag{48}$$

Thus, the Einstein's equations in the frequency domain can be obtained as

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) h'_{\mu\nu}(\omega, x^\alpha) = -KT_{\mu\nu}(\omega, x^\alpha) \tag{49}$$

Assume that the source of the GW is confined in a region $|x^\alpha| \leq \varepsilon$, which is much smaller than the wavelength of the emitted GW, namely,

$$\lambda_{GW} = \frac{2\pi c}{\omega} \gg \varepsilon$$

which leads to a slow-motion approximation

$$v = \omega\epsilon \ll c \tag{50}$$

To solve (49), the equations are integrated inside and outside the source region, respectively, then the two solutions are matched on the boundary of the source region.

A. Wave with Isotropic Source

At first, consider a solution of a point source, which is independent of ϕ and θ. Outside the source region, (49) can be represented in the spherical coordinate as

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial h'_{\mu\nu}}{\partial r} + \frac{\omega^2}{c^2} h'_{\mu\nu} = 0 \tag{51}$$

of which the solution can be represented as

$$h'_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r} e^{i\omega r/c} + \frac{B_{\mu\nu}(\omega)}{r} e^{-i\omega r/c} \tag{52}$$

By causality, only the wave emitted from the source is accepted, thus (52) is reduced to

$$h'_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r} e^{i\omega r/c} \tag{53}$$

Inside the source region, by integrating each term in (49) over the source volume V_ε, we have

$$\int_{V_\varepsilon}\left(\nabla^2+\frac{\omega^2}{c^2}\right)h'_{\mu\nu}(\omega,x^\alpha)dV$$
$$=-K\int_{V_\varepsilon}T_{\mu\nu}(\omega,x^\alpha)dV \tag{54}$$

Next, the left-hand side of (54) can be reduced to

$$\int_{V_\varepsilon}\nabla^2 h'_{\mu\nu}dV=\int_{V_\varepsilon}\nabla\cdot\nabla h'_{\mu\nu}dV$$
$$=\int_{S_\varepsilon}\nabla h'_{\mu\nu}dS\simeq 4\pi\varepsilon^2\left.\frac{dh'_{\mu\nu}}{dr}\right|_{r=\varepsilon} \tag{55}$$

By substituting (53) into (55), we have

$$\int_{V_\varepsilon}\nabla^2 h'_{\mu\nu}dV\simeq 4\pi\varepsilon^2\left.\frac{dh'_{\mu\nu}}{dr}\right|_{r=\varepsilon}$$
$$=4\pi\varepsilon^2\left(\frac{d}{dr}\frac{A_{\mu\nu}(\omega)}{r}e^{i\omega r/c}\right)_{r=\varepsilon}$$
$$=4\pi\varepsilon^2\left(-\frac{A_{\mu\nu}}{r^2}e^{i\omega r/c}+\frac{A_{\mu\nu}}{r}\frac{i\omega}{c}e^{i\omega r/c}\right)_{r=\varepsilon}$$
$$\simeq -4\pi A_{\mu\nu}(\omega) \tag{56}$$

Then, by substituting (56) into (54), we obtain

$$-4\pi A_{\mu\nu}(\omega)+\int_{V_\varepsilon}\frac{\omega^2}{c^2}h'_{\mu\nu}(\omega,x^\alpha)dV$$
$$=-K\int_{V_\varepsilon}T_{\mu\nu}(\omega,x^\alpha)dV \tag{57}$$

Since

$$\int_{V_\varepsilon}\frac{\omega^2}{c^2}h'_{\mu\nu}(\omega,x^\alpha)dV<\frac{\omega^2}{c^2}\left|h'_{\mu\nu}\right|_{\max}\frac{4}{3}\pi\varepsilon^3$$

which is negligible, (57) can be reduced to

$$-4\pi A_{\mu\nu}(\omega)=-K\int_{V_\varepsilon}T_{\mu\nu}(\omega,x^\alpha)dV \tag{58}$$

leading to

$$A_{\mu\nu}(\omega)=\frac{4G}{c^4}\int_{V_\varepsilon}T_{\mu\nu}(\omega,x^\alpha)dV \tag{59}$$

Finally, by substituting (59) into (53), we have

$$h'_{\mu\nu}(\omega,r)=\frac{4G}{c^4 r}e^{i\omega r/c}\int_{V_\varepsilon}T_{\mu\nu}(\omega,x^\alpha)dV$$

which can be transformed to the time domain as

$$h'_{\mu\nu}(t,r)=\frac{4G}{c^4 r}\int_{V_\varepsilon}T_{\mu\nu}(t-r/c,x^\alpha)dV \tag{60}$$

A second-rank tensor can be represented as

$$\bar{\bar{T}}=T^{\alpha\beta}\bar{e}_\alpha\bar{e}_\beta=T_{\alpha\beta}\bar{e}^\alpha\bar{e}^\beta \tag{61}$$

where

$$T_{\alpha\beta}=g_{\alpha\alpha'}g_{\beta\beta'}T^{\alpha'\beta'}$$

$\bar{e}_\alpha$ and $\bar{e}^\alpha$ are dual vectors, serving as the basis vectors. If $g_{\alpha\beta}$ represents a Minkowski metric, we have $T_{\alpha\beta}=T^{\alpha\beta}$. In this case, any equation represented in covariant form or contravariant form will be the same, the only difference is the coordinate system or basis vectors chosen. Hence, (60) implies the contravariant form as

$$h'^{\mu\nu}(t,r)=\frac{4G}{c^4 r}\int_{V_\varepsilon}T^{\mu\nu}(t-r/c,x^\alpha)dV \tag{62}$$

Consider the conservation law of a flat spacetime

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu}=0 \tag{63}$$

which is decomposed into space and time components as

$$\frac{1}{c}\frac{\partial T^{\mu 0}}{\partial t}=-\frac{\partial T^{\mu\alpha}}{\partial x^\alpha} \tag{64}$$

By integrating (64) over the source volume V_ε, we have

$$\int_{V_\varepsilon}\frac{1}{c}\frac{\partial T^{\mu 0}}{\partial t}dV=-\int_{V_\varepsilon}\frac{\partial T^{\mu k}}{\partial x^\alpha}dV$$
$$=-\int_S T^{\mu\alpha}dS_\alpha \tag{65}$$

On the surface S,

$$T^{\mu\nu}=0 \tag{66}$$

Thus, (65) is reduced to

$$\frac{1}{c}\frac{\partial}{\partial t}\int_{V_\varepsilon}T^{\mu 0}dV=0 \tag{67}$$

Next, multiplying (64) by x^k and integrating over the source volume V_ε, we have

$$\frac{1}{c}\frac{\partial}{\partial t}\int_{V_\varepsilon}T^{\mu 0}x^k dV=-\int_{V_\varepsilon}\frac{\partial T^{\mu\alpha}}{\partial x^\alpha}x^k dV$$
$$=-\left[\int_{V_\varepsilon}\frac{\partial T^{\mu\alpha}x^k}{\partial x^\alpha}dV-\int_{V_\varepsilon}T^{\mu\alpha}\frac{\partial x^k}{\partial x^\alpha}dV\right]$$
$$=-\int_S T^{\mu\alpha}x^k dS_\alpha+\int_{V_\varepsilon}T^{\mu k}dV=\int_{V_\varepsilon}T^{\mu k}dV \tag{68}$$

where the last step is derived by imposing $T^{\mu\nu}=0$ on S, as mentioned in (66). Since $T^{\mu k}$ is symmetric, (68) can be rewritten as

$$\frac{1}{2c}\frac{\partial}{\partial t}\int_{V_\varepsilon}\left(T^{\mu 0}x^k+T^{k0}x^\mu\right)dV=\int_{V_\varepsilon}T^{\mu k}dV \tag{69}$$

Next, consider (64) with $\mu=0$,

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t}+\frac{\partial T^{0\alpha}}{\partial x^\alpha}=0 \tag{70}$$

By multiplying (70) with $x^k x^n$ and integrating over the source volume V_ε, we have

$$\frac{1}{c}\frac{\partial}{\partial t}\int_{V_\varepsilon}T^{00}x^k x^n dV=-\int_{V_\varepsilon}\frac{\partial T^{0\alpha}}{\partial x^\alpha}x^k x^n dV$$
$$=-\left[\int_{V_\varepsilon}\frac{\partial T^{0\alpha}x^k x^n}{\partial x^\alpha}dV\right.$$
$$\left.-\int_{V_\varepsilon}T^{0\alpha}\left(\frac{\partial x^k}{\partial x^\alpha}x^n+x^k\frac{\partial x^n}{\partial x^\alpha}\right)dV\right]$$
$$=-\int_S T^{0\alpha}x^k x^n dS+\int_{V_\varepsilon}\left(T^{0k}x^n+T^{0n}x^k\right)dV$$
$$=\int_{V_\varepsilon}\left(T^{0k}x^n+T^{0n}x^k\right)dV \tag{71}$$

where $T^{\mu\nu} = 0$ on S.

By differentiating (71) with respect to $x^0 = ct$, we obtain

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_{V_\varepsilon} T^{00}x^k x^n dV \\ &= \frac{1}{c}\frac{\partial}{\partial t}\int_{V_\varepsilon}\left(T^{0k}x^n + T^{0n}x^k\right) dV \end{aligned} \tag{72}$$

By substituting (69) into (72), we have

$$\int_{V_\varepsilon} T^{kn} dV = \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{V_\varepsilon} T^{00}x^k x^n dV \tag{73}$$

Define the quadrupole moment tensor of the system as

$$q^{kn}(t) = \frac{1}{c^2}\int_{V_\varepsilon} T^{00}(t, x^\alpha) x^k x^n dV \tag{74}$$

which is a function of time only. Thus, (73) can be rewritten as

$$\int_{V_\varepsilon} T^{kn} dV = \frac{1}{2}\frac{d^2}{dt^2} q^{kn}(t) \tag{75}$$

From (67), we have

$$\int_{V_\varepsilon} T^{\mu 0} dV = \text{const}$$

Thus, comparing with (62), we have

$$h'^{\mu 0} = \text{const}$$

Since we are only interested in the time-dependent part of the field, we let

$$h'^{\mu 0} = 0 \tag{76}$$

By (76) and substituting (75) into (62), $h'(t, r)$ can be represented as

$$\begin{aligned} h'^{\mu 0} &= 0, \qquad \mu = 0, 1, 2, 3 \\ h'^{kn} &= \frac{2G}{c^4 r}\frac{d^2}{dt^2} q^{kn}(t - r/c), \qquad k, n = 1, 2, 3 \end{aligned} \tag{77}$$

By the same argument after (61), the covariant form of (77) is

$$\begin{aligned} h'_{\mu 0} &= 0, \qquad \mu = 0, 1, 2, 3 \\ h'_{kn} &= \frac{2G}{c^4 r}\frac{d^2}{dt^2} q^{kn}(t - r/c), \qquad k, n = 1, 2, 3 \end{aligned} \tag{78}$$

A general gravitational plane-wave (GPW) propagating in the z-direction can be represented with a (covariant) perturbation tensor

$$h_{\mu\nu}(t, z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(u) & h_\times(u) & 0 \\ 0 & h_\times(u) & -h_+(u) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{79}$$

where $h_+(u)$ and $h_\times(u)$ are the amplitudes of the $+$-polarized and $\times$-polarized GPW, respectively. Eqn.(79) indicates that the polarization of the GPW is transverse to its propagation direction, namely,

$$n^\alpha h_{\alpha\beta} = 0 \tag{80}$$

where n^α represents the propagation direction. The sum of the diagonal components in $h_{\mu\nu}(t, z)$ is zero (traceless), namely,

$$\delta^{\alpha\beta} h_{\alpha\beta} = 0 \tag{81}$$

Define an operator which projects a vector onto the plane transverse to the propagation direction, $\hat{n}$, as

$$P_{\alpha\beta} = \delta_{\alpha\beta} - n_\alpha n_\beta \tag{82}$$

Then, define a transverse-traceless projector as

$$P_{\alpha\beta\gamma\lambda} = P_{\alpha\gamma}P_{\beta\lambda} - \frac{1}{2}P_{\alpha\beta}P_{\gamma\lambda} \tag{83}$$

which extracts the transverse-traceless (TT) part of a second-rank tensor. Thus, the GW and the quadrupole moment projected with the TT projector become

$$\begin{aligned} h_{\alpha\beta}^{TT} &= P_{\alpha\beta\gamma\lambda} h_{\gamma\lambda} = P_{\alpha\beta\gamma\lambda} h'_{\gamma\lambda} = h'^{TT}_{\alpha\beta} \\ q_{\alpha\beta}^{TT} &= P_{\alpha\beta\gamma\lambda} q_{\gamma\lambda} \end{aligned} \tag{84}$$

$P_{\alpha\beta\gamma\lambda}$ also have the properties

$$\begin{aligned} &P_{\alpha\beta\gamma\lambda}P_{\gamma\lambda mn} = P_{\alpha\beta mn} \\ &n^\alpha P_{\alpha\beta\gamma\lambda} = n^\beta P_{\alpha\beta\gamma\lambda} = n^\gamma P_{\alpha\beta\gamma\lambda} = n^\delta P_{\alpha\beta\gamma\delta} = 0 \\ &\delta^{\alpha\beta} P_{\alpha\beta\gamma\lambda} = \delta^{\gamma\delta} P_{\alpha\beta\gamma\lambda} \end{aligned} \tag{85}$$

B. Wave with Binary Source

Consider a binary system of two celestial bodies, B_1 and B_2, with masses m_1 and m_2, respectively, moving on a planar (xy) coordinate system with the origin coincident with the center of mass. The distances of B_1 and B_2 from the center of mass are a_1 and a_2, respectively. Define

$$a = a_1 + a_2, \qquad M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{86}$$

which imply

$$a_1 = \frac{m_2 a}{M}, \qquad a_2 = \frac{m_1 a}{M} \tag{87}$$

By the Newton's gravitation law, we have

$$\frac{G m_1 m_2}{a^2} = m_1 \omega^2 \frac{m_2 a}{M}$$

where

$$\omega = \sqrt{\frac{GM}{a^3}} \tag{88}$$

is the orbital angular frequency. Without loss of generality, the orbits of B_1 and B_2 can be represented as

$$\begin{aligned} (x_1, y_1) &= \left(\frac{m_2 a}{M}\cos(\omega t), \frac{m_2 a}{M}\sin(\omega t)\right) \\ (x_2, y_2) &= \left(-\frac{m_1 a}{M}\cos(\omega t), -\frac{m_1 a}{M}\sin(\omega t)\right) \end{aligned} \tag{89}$$

The energy-momentum tensor of this binary system is [4]

$$T^{00} = \sum_{n=1}^{2} m_n c^2 \delta(x - x_n)\delta(y - y_n)\delta(z) \tag{90}$$

By substituting (90) into (74), the quadrupole-moment tensor is derived as

$$\begin{aligned} q_{xx} &= m_1 x_1^2 + m_2 x_2^2 \\ q_{yy} &= m_1 y_1^2 + m_2 y_2^2 \\ q_{xy} &= q_{yx} = m_1 x_1 x_2 + m_2 x_1 x_2 \end{aligned} \tag{91}$$

By substituting the orbital coordinates in (89) into (91), we have

$$\begin{aligned} q_{xx} &= \mu a^2 \cos^2(\omega t) \\ q_{yy} &= \mu a^2 \sin^2(\omega t) \\ q_{xy} &= q_{yx} = \mu a^2 \cos(\omega t)\sin(\omega t) \end{aligned} \tag{92}$$

By applying the definition in (86), the time-dependent components of (92) can be calculated as

$$\begin{aligned} q_{xx} &= -q_{yy} = \frac{1}{2}\mu a^2 \cos(2\omega t) \\ q_{xy} &= q_{yx} = \frac{1}{2}\mu a^2 \sin(2\omega t) \end{aligned} \tag{93}$$

Consider a GW propagating in the z direction, or $\hat{n} = (0,0,1)$. The GW can be represented, by using (78), as

$$\begin{aligned} h_{\mu 0}^{TT} &= 0, \qquad \mu = 0,1,2,3 \\ h_{mn}^{TT}(t,z) &= \frac{2G}{c^4 z}\frac{d^2}{dt^2} q_{mn}^{TT}(t - z/c), \qquad m,n = 1,2,3 \end{aligned} \tag{94}$$

where

$$q_{mn}^{TT} = P_{\alpha\beta mn} q_{mn} \tag{95}$$

from the relation in (84). Then, by substituting (83) into (95), q_{kn}^{TT} can be calculated as

$$\begin{aligned} q_{xx}^{TT} &= \left(P_{x\gamma}P_{x\delta} - \frac{1}{2}P_{xx}P_{\gamma\delta}\right) q_{\gamma\delta} \\ &= \left(P_{xx}P_{xx} - \frac{1}{2}P_{xx}^2\right) q_{xx} - \frac{1}{2}P_{xx}P_{yy}q_{yy} \\ &= \frac{1}{2}(q_{xx} - q_{yy}) \\ q_{yy}^{TT} &= \left(P_{y\gamma}P_{y\delta} - \frac{1}{2}P_{yy}P_{\gamma\delta}\right) q_{\gamma\delta} \\ &= -\frac{1}{2}(q_{xx} - q_{yy}) \\ q_{xy}^{TT} &= \left(P_{x\gamma}P_{y\delta} - \frac{1}{2}P_{xy}P_{\gamma\delta}\right) q_{\gamma\delta} \\ &= P_{xx}P_{yy}q_{xy} = q_{xy} \end{aligned} \tag{96}$$

and the remaining components are all zeros. Summarizing from (80), (94) and (96), a GW propagating in the z direction can be represented as

$$\begin{aligned} h_{\mu 0}^{TT} &= 0, \qquad h_{z\alpha}^{TT} = 0, \\ h_{xx}^{TT} &= -h_{yy}^{TT} = \frac{G}{c^4 z}\frac{d^2}{dt^2}(q_{xx} - q_{yy}), \\ h_{xy}^{TT} &= \frac{2G}{c^4 z}\frac{d^2}{dt^2} q_{xy} \end{aligned} \tag{97}$$

Finally, by substituting (93) into (97), we have

$$\begin{aligned} h_{xx}^{TT} &= -h_{yy}^{TT} = -\frac{G}{c^4 z}\mu a^2 (2\omega)^2 \cos[2\omega(t - z/c)] \\ h_{xy}^{TT} &= -\frac{G}{c^4 z}\mu a^2 (2\omega)^2 \sin[2\omega(t - z/c)] \end{aligned} \tag{98}$$

which implies that the GW is circularly polarized and its frequency is twice that of the orbital frequency.

The rate at which energy is carried away by a GW, as represented in (94), is [5]

$$\frac{dE_{\rm GW}}{dt} = \frac{32G}{5c^5}\mu^2 a^4 \omega^6 \tag{99}$$

which is derived in the Appendix. By substituting (88) into (99), we have another form

$$\frac{dE_{\rm GW}}{dt} = \frac{32G^4\mu^2 M^3}{5c^5 a^5} \tag{100}$$

Using Newton's law of motion and gravitation, with the relation in(86) and (87), the orbital kinetic energy of a binary system is

$$\begin{aligned} E_k &= \frac{1}{2}m_1(\omega a_1)^2 + m_2(\omega a_2)^2 \\ &= \frac{1}{2}\omega^2 \left[m_1\left(\frac{m_2 a}{M}\right)^2 + m_2\left(\frac{m_1 a}{M}\right)^2\right] \\ &= \frac{1}{2}\omega^2 \mu a^2 = \frac{GM\mu}{2a} \end{aligned} \tag{101}$$

and the orbital potential energy is

$$U = -\frac{Gm_1 m_2}{a} = -\frac{GM\mu}{a} \tag{102}$$

By summing (101) and (102), we obtain the total orbital energy

$$E_{\rm orb} = -\frac{GM\mu}{2a} \tag{103}$$

implying that

$$-\frac{dE_{\rm orb}}{dt} = -\frac{GM\mu}{2a^2}\left(\frac{da}{dt}\right) \tag{104}$$

which can be reduced to

$$\frac{1}{a}\frac{da}{dt} = -\frac{1}{E_{\rm orb}}\frac{dE_{\rm orb}}{dt} \tag{105}$$

By substituting (100) and (103) into (105), we have

$$\frac{1}{a}\frac{da}{dt} = -\frac{64G^3\mu M^2}{5c^5}\frac{1}{a^4} \tag{106}$$

which can be integrated to obtain

$$a^4(t) = a_0^4 - \left(\frac{256G^3\mu M^2}{5c^5}\right)t = a_0^4\left(1 - \frac{256G^3\mu M^2}{5c^5 a_0^4}t\right)$$

or

$$a(t) = a_0\left(1 - \frac{t}{t_c}\right)^{1/4}$$

where a_0 is the orbital separation at $t = 0$, and

$$t_c = \frac{5c^5 a_0^4}{256G^3\mu M^2} \tag{107}$$

which indicates the order of magnitude of the time the system takes to merge, starting from a given distance a_0.

By substituting (88) into (107), we have

$$\omega(t) = \sqrt{\frac{GM}{a_0^3(1 - t/t_c)^{3/4}}} = \frac{\omega_0}{(1 - t/t_c)^{3/8}} \tag{108}$$

where

$$\omega_0 = \sqrt{\frac{GM}{a_0^3}}$$

The GW frequency $\nu(t)$ is twice the orbital frequency, thus can be be derived from (108) as

$$\nu(t) = \frac{\nu_0}{(1 - t/t_c)^{3/8}} \tag{109}$$

where

$$\nu_0 = \frac{1}{\pi}\sqrt{\frac{GM}{a_0^3}}$$

is the GW frequecy at a_0.

From (98), the instantaneous amplitude of the GW can be derived as

$$\begin{aligned} h_0(t) &= \frac{4G\mu a^2\omega^2}{zc^4} = \frac{4G^2\mu M}{zc^4 a} \\ &= \frac{4G^2\mu M}{zc^4}\frac{\omega^{2/3}(t)}{(GM)^{1/3}} = \frac{4G^{5/3}\mu M^{2/3}}{zc^4}\omega^{2/3}(t) \end{aligned} \tag{110}$$

where the last three steps are derived by using (88) and $a_0 = (GM/\omega^2)^{1/3}$. Define a chirp mass M_c as

$$M_c = (\mu^3 M^2)^{1/5} \tag{111}$$

Then, (110) can be reduced to

$$h_0(t) = \frac{4\pi^{2/3}G^{5/3}M_c^{5/3}}{zc^4}\nu^{2/3}(t) \tag{112}$$

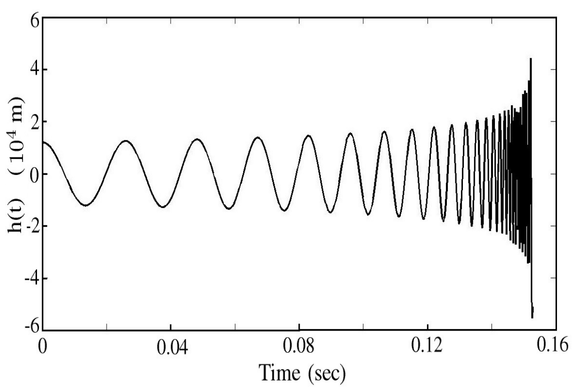

Fig. 9. Simulated gravitational waveform of GW150914, $z = 1$, $m_1 = m_2 = 30M_\odot$ and $a_0 = 900$ km.

Fig.9 shows the simulated gravitational waveform of GW150914. As the two bodies move closer, both the GW frequency and amplitude increase.

From the relation in (88), a and da/dt can be represented as

$$\begin{aligned} a &= \left(\frac{GM}{\omega^2}\right)^{1/3} \\ \frac{da}{dt} &= -\frac{2}{3}(GM)^{1/3}\omega^{-5/3}\left(\frac{d\omega}{dt}\right) \end{aligned} \tag{113}$$

Since $dE_{\mathrm{GW}}/dt = -dE_{\mathrm{orb}}/dt$, by equating (99) and (104), and then replacing all the a and da/dt with (113), we derive

$$\begin{aligned} \frac{d\omega}{dt} &= \frac{96}{5c^5}\omega^{11/3}G^{5/3}\mu M^{2/3} \\ &= \frac{96\omega^{11/3}}{5c^5}(GM_c)^{5/3} \\ &= c_1 f(\tilde{m}_1, \tilde{m}_2) \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{96G^{5/3}}{5c^5}\omega^{11/3} = 8.67 \times 10^{-59}\omega^{11/3} \\ f(\tilde{m}_1, \tilde{m}_2) &= \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \\ \omega_0 &= \sqrt{\frac{GM}{a^3}} \end{aligned}$$

In general relativity, the field equations governing spacetime curvature are nonlinear. Therefore, it is difficult to solve in a closed form. Schwarzschild solution is an adequate approximation only when the mass of one star is overwhelming greater than the mass of the other, such as a photon passing a star or a planet orbiting its sun. In a binary star system, the metric for the case of two comparable masses cannot be solved in closed form, so we have to resort to approximation techniques such as the post-Newtonian approximation or numerical approximations.

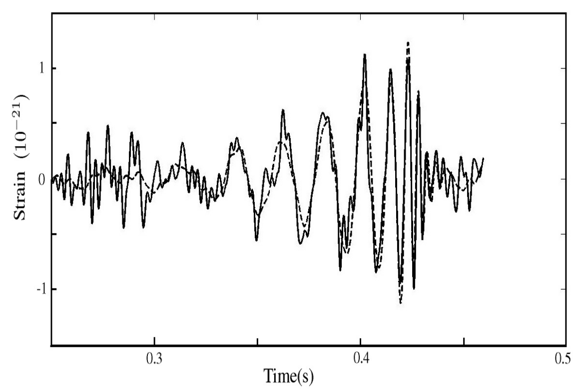

Fig. 10. The gravitational wave event GW150914 detected by LIGO Hanford detector in the 35-350 Hz band (——–) and the reconstructed waveform by numerical relativity (− − −) [2].

The GW150914 observation included eight gravitational-wave cycles(about 0.2 second), covering the late inspiral, merger, and ringdown phases of the binary. This late phase of a BBH merger can be described accurately only by directly solving the full equations of general relativity, because analytic approximations fail near the time of merger [3]. In Fig.10, the solid line shows the gravitational wave signal detected by LIGO Hanford detector on September 14th, 2015, filtered with a 35 - 350 Hz bandpass filter. The dash line shows a reconstruction of waveform by numerical relativity.

According to general relativity, two masses orbiting one another will emit gravitational wave, causing the orbits to gradually lose energy, which will be transpoted away by these waves. Unlike electromagnetic waves which can be emitted by a dipole source, gravitational waves are emitted by quadrupoles at least. Thus, the gravitational field is a tensor rather than vector field.

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Appendix I: GW Luminosity

To study the energy-momentum tensor carried by GWs, we must extend the approximation from (43) to [6]

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \tag{114}$$

where $g_{\mu\nu}^{(0)}$ is the background (zero-order) Minkowski metric, and $h_{\mu\nu}$ is the first-order perturbation associated with the GW. Thus, the Ricci tensor can be represented as

$$R_{\mu\nu} = R_{\mu\nu}^{(0)}(g_{\mu\nu}^{(0)}) + R_{\mu\nu}^{(1)}(h_{\mu\nu}) + R_{\mu\nu}^{(2)}(h_{\mu\nu}h_{\lambda m})$$

The short-wave approximation assumes that the scale on which $h_{\mu\nu}$ varies (d) are much smaller than the scale on which $g_{\mu\nu}^{(0)}$ varies (D). By introducing a length scale S, with $D \gg S \gg d$, the background effect is separated from the perturbation. Next, define the average of a variable α, denoted as $\langle\alpha\rangle_s$, over a spatial volume of side length S. This averaging scheme renders a physical quantity that varies on the scale of d to zero, while that varies on the scale of D to a constant.

The Einstein equation is represented as [4]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The energy-momentum tensor of a GW reduces to

$$T_{\mu\nu}^{(\rm GW)} = -\frac{c^4}{8\pi G}\langle R_{\mu\nu}^{(2)} - \frac{1}{2}R^{(2)}g_{\mu\nu}^{(0)}\rangle_s \tag{115}$$

where $T_{\mu\nu}^{(0)} = T_{\mu\nu}^{(1)} = 0$ and $T_{\mu\nu}^{(\rm GW)}$ is of second order. The Ricci tensor is defined as [4]

$$\begin{aligned} R_{\mu\nu} &= R_{\mu\lambda\nu}^{\lambda} \\ &= \left(\frac{\partial\Gamma_{\mu\nu}^{\lambda}}{\partial x^{\lambda}} + \Gamma_{m\lambda}^{\lambda}\Gamma_{\mu\nu}^{m}\right) - \left(\frac{\partial\Gamma_{\mu\lambda}^{\lambda}}{\partial x^{\nu}} + \Gamma_{m\nu}^{\lambda}\Gamma_{\mu\lambda}^{m}\right) \end{aligned} \tag{116}$$

with

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda} &= \frac{1}{2}g^{\lambda\ell}\left(\frac{\partial g_{\ell\nu}}{\partial x^{\mu}} + \frac{\partial g_{\ell\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\ell}}\right) \\ &= \frac{1}{2}g^{\lambda\ell}(g_{\ell\nu,\mu} + g_{\ell\mu,\nu} - g_{\mu\nu,\ell}) \end{aligned} \tag{117}$$

In the contravariant notation, the metric tensor g is represented as

$$g^{\mu\nu} = g^{\mu\nu(0)} + \tilde{h}^{\mu\nu} \tag{118}$$

Since

$$\begin{aligned} \delta_{\nu}^{\mu} &= g^{\mu\lambda}g_{\lambda\nu} = \left(g^{\mu\lambda(0)} + \tilde{h}^{\mu\lambda}\right)\left(g_{\lambda\nu}^{(0)} + h_{\lambda\nu}\right) \\ &= \delta_{\nu}^{\mu} + h_{\nu}^{\mu} + \tilde{h}_{\nu}^{\mu} \end{aligned}$$

which implies that

$$\tilde{h}_{\nu}^{\mu} = -h_{\nu}^{\mu} \tag{119}$$

which is multiplied with $g^{\nu\nu}$ on the both sides to derive

$$\tilde{h}^{\mu\nu} = -h^{\mu\nu} \tag{120}$$

Next, by substituting (114), (118) and (120) into (117), the Christoffel symbol can be expanded to different orders as

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda(0)} &= \frac{1}{2}g^{\lambda\ell(0)}\left(g_{\ell\nu,\mu}^{(0)} + g_{\ell\mu,\nu}^{(0)} - g_{\mu\nu,\ell}^{(0)}\right) \\ \Gamma_{\mu\nu}^{\lambda(1)} &= -\frac{1}{2}h^{\lambda\ell}\left(g_{\ell\nu,\mu}^{(0)} + g_{\ell\mu,\nu}^{(0)} - g_{\mu\nu,\ell}^{(0)}\right) \\ &\quad + \frac{1}{2}g^{\lambda\ell(0)}\left(h_{\ell\nu,\mu} + h_{\ell\mu,\nu} - h_{\mu\nu,\ell}\right) \\ \Gamma_{\mu\nu}^{\lambda(2)} &= \frac{1}{2}h^{\lambda\rho}h_{\rho}^{\ell}\left(g_{\ell\nu,\mu}^{(0)} + g_{\ell\mu,\nu}^{(0)} - g_{\mu\nu,\ell}^{(0)}\right) \\ &\quad - \frac{1}{2}h^{\lambda\ell}\left(h_{\ell\nu,\mu} + h_{\ell\mu,\nu} - h_{\mu\nu,\ell}\right) \end{aligned} \tag{121}$$

Note that the comma in the subscript as in $,\nu$ indicates a partial derivative with respect to x_{ν}. Similarly, the Ricci tensor $R_{\mu\nu}$ in (116) can be expanded to different orders as

$$\begin{aligned} R_{\mu\nu}^{(0)} &= \frac{1}{2}g^{\lambda\ell(0)}\left(g_{\ell\nu,\mu\lambda}^{(0)} + g_{\mu\lambda,\ell\nu}^{(0)} - g_{\ell\lambda,\mu\nu}^{(0)} - g_{\mu\nu,\ell\lambda}^{(0)}\right) \\ R_{\mu\nu}^{(1)} &= \frac{1}{2}g^{\lambda\ell(0)}\left(h_{\ell\nu,\mu\lambda} + h_{\mu\lambda,\ell\nu} - h_{\ell\lambda,\mu\nu} - h_{\mu\nu,\ell\lambda}\right) \\ &\quad - \frac{1}{2}h^{\lambda\ell}\left(g_{\ell\nu,\mu\lambda}^{(0)} + g_{\mu\lambda,\ell\nu}^{(0)} - g_{\ell\lambda,\mu\nu}^{(0)} - g_{\mu\nu,\ell\lambda}^{(0)}\right) \\ R_{\mu\nu}^{(2)} &= -\frac{1}{2}h^{\lambda\ell}\left(h_{\ell\nu,\mu\lambda} + h_{\mu\lambda,\ell\nu} - h_{\ell\lambda,\mu\nu} - h_{\mu\nu,\ell\lambda}\right) \\ &\quad + \frac{1}{2}h^{\lambda\ell}h_{\rho}^{\ell}\left(g_{\ell\nu,\mu\lambda}^{(0)} + g_{\mu\lambda,\ell\nu}^{(0)} - g_{\ell\lambda,\mu\nu}^{(0)} - g_{\mu\nu,\ell\lambda}^{(0)}\right) \end{aligned} \tag{122}$$

The Ricci tensor in (116) can be rewritten as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}g^{\lambda\ell}\left(g_{\ell\nu,\mu\lambda} + g_{\mu\lambda,\ell\nu} - g_{\ell\lambda,\mu\nu} - g_{\mu\nu,\ell\lambda}\right) \\ &\quad + g^{\lambda\ell}g_{m\rho}\left(\Gamma_{\mu\nu}^{m}\Gamma_{\ell\lambda}^{\rho} - \Gamma_{\mu\lambda}^{m}\Gamma_{\nu\ell}^{\rho}\right) \end{aligned} \tag{123}$$

Then, by substituting (121) into (123) and comparing with (122), The second-order terms can be represented as

$$\begin{aligned} R_{\mu\nu}^{(2)} &= -\frac{1}{2}h^{\lambda\ell}\left(h_{\ell\nu,\mu\lambda} + h_{\mu\lambda,\ell\nu} - h_{\ell\lambda,\mu\nu} - h_{\mu\nu,\ell\lambda}\right) \\ &\quad + \frac{1}{4}g^{\lambda\ell(0)}g_{m\rho(0)}\Big[\left(h_{\mu,\nu}^{m} + h_{\nu,\mu}^{m} - h_{\mu\nu}^{;m}\right)\left(h_{\lambda,\ell}^{\rho} + h_{\ell,\lambda}^{\rho} - h_{\ell\lambda}^{;\rho}\right) \\ &\quad - \left(h_{\mu,\lambda}^{m} + h_{\lambda,\mu}^{m} - h_{\mu\lambda}^{;m}\right)\left(h_{\nu,\ell}^{\rho} + h_{\ell,\nu}^{\rho} - h_{\nu\ell}^{;\rho}\right)\Big] \end{aligned} \tag{124}$$

The second trem on the right-hand side of (124) is further reduced to

$$g^{\lambda\ell(0)}g_{m\rho(0)}(h_{\lambda,\ell}^{\rho} + h_{\ell,\lambda}^{\rho} - h_{\ell\lambda}^{;\rho}) = h_{m,\ell}^{\ell} + h_{m,\lambda}^{\lambda} - h_{,m} = 0$$

where the gauge condition in (44)

$$h^{\lambda}_{\mu,\lambda} - \frac{1}{2}h_{,\mu} = 0 \tag{125}$$

is used. Thus, (124) is reduced to

$$\begin{aligned}
R^{(2)}_{\mu\nu} &= -\frac{1}{2}h^{\lambda\ell}\left(h_{\ell\nu,\mu\lambda} + h_{\mu\lambda,\ell\nu} - h_{\ell\lambda,\mu\nu} - h_{\mu\nu,\ell\lambda}\right) \\
&-\frac{1}{4}g^{\lambda\ell(0)}g_{m\rho(0)}\left[\left(h^{m}_{\mu,\lambda} + h^{m}_{\lambda,\mu} - h^{;m}_{\mu\lambda}\right)\left(h^{\rho}_{\nu,\ell} + h^{\rho}_{\ell,\nu} - h^{;\rho}_{\nu\ell}\right)\right] \\
&= -\frac{1}{2}h^{\lambda\ell}\left(h_{\ell\nu,\mu\lambda} + h_{\mu\lambda,\ell\nu} - h_{\ell\lambda,\mu\nu} - h_{\mu\nu,\ell\lambda}\right) \\
&-\frac{1}{4}\left(h^{m,\ell}_{\mu} + h^{m\ell}_{,\mu} - h^{\ell,m}_{\mu}\right)\left(h_{m\nu,\ell} + h_{\ell m,\nu} - h_{\nu\ell,m}\right)
\end{aligned} \tag{126}$$

Then, the second-order Ricci scalar is calculated as

$$\begin{aligned}
R^{(2)} &= g^{\mu\nu(0)}R^{(2)}_{\mu\nu} \\
&= -\frac{1}{2}h^{\lambda\ell}\left[h^{\mu}_{\ell,\mu\lambda} + h^{\nu}_{\lambda,\nu\ell} - \left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)h_{\ell\lambda} - h_{,\ell\lambda}\right] \\
&\quad -\frac{1}{4}\left(h^{m\nu,\ell} + h^{m\ell,\nu} - h^{\ell\nu,m}\right)\left(h_{m\nu,\ell} + h_{m\ell,\nu} - h_{\ell\nu,m}\right)
\end{aligned} \tag{127}$$

By substituting (126) into (127) and using the gauge condition in (125),we derive

$$\begin{aligned}
&\frac{1}{2}R^{(2)}g^{(0)}_{\mu\nu} = -\frac{1}{4}g^{(0)}_{\mu\nu}\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)h \\
&+\frac{1}{8}g^{(0)}_{\mu\nu}(h^{m\nu,\ell} + h^{m\ell,\nu} - h^{\ell\nu,m})(h_{m\nu,\ell} + h_{m\ell,\nu} - h_{\ell\nu,m})
\end{aligned} \tag{128}$$

Then, by subtracting (126) with (128), we have

$$\begin{aligned}
&R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(2)}g^{(0)}_{\mu\nu} = \frac{1}{2}\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)h'_{\mu\nu} \\
&-\frac{1}{4}(h_{m\nu,\ell} + h_{m\ell,\nu} - h_{\ell\nu,m})(h^{m\nu,\ell} + h^{m\ell}_{,\mu} - h^{\ell,m}_{\mu}) \\
&+\frac{1}{8}(h_{m\nu,\ell} + h_{m\ell,\nu} - h_{\ell\nu,m})(h^{m\nu,\ell} + h^{m\ell,\nu} - h^{\ell\nu,m}) \\
&= \frac{1}{4}h_{m\ell,\nu}h^{m\ell}_{,\mu} - \frac{1}{8}h_{,\nu}h^{,\nu}
\end{aligned} \tag{129}$$

where the last step is reduced by imposing the field equation $\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)h'_{\mu\nu} = 0$ and $h_{\mu 0} = h_{\mu z} = 0$ from (79).

Finally, by substituting (129) into (115), the energy-momentum tensor of GWs becomes

$$T_{00} = \frac{c^2}{32\pi G}\sum_{\alpha\beta}\left(\frac{dh^{TT}_{\alpha\beta}}{dt}\right)^2$$

where $h^{TT}_{\alpha\beta}$ is the transverce-traceless part of $h^{\alpha\beta}$, leading to $h_{\mu 0} = h_{\mu z} = 0$ in (79). Since the energy of the gravitational field cannot be defined locally in general relativity, we need to average over several wavelengths to find the GW-flux

$$\frac{dE_{GW}}{dtdS} = \langle cT_{\mu\nu}\rangle = \frac{c^3}{32\pi G}\left\langle\sum_{\alpha\beta}\left(\frac{dh^{TT}_{\alpha\beta}}{dt}\right)^2\right\rangle \tag{130}$$

By substituting h_{kn} in (94) into (130) and applying (84), we have

$$\begin{aligned}
\frac{dE_{GW}}{dtdS} &= \frac{G}{8\pi c^5 r^2}\left\langle\sum_{\alpha\beta}\left(\frac{d^3 q^{TT}_{\alpha\beta}}{dt^3}\right)^2\right\rangle \\
&= \frac{G}{8\pi c^5 r^2}\left\langle\sum_{\alpha\beta}\left(\frac{d^3(P_{mn\alpha\beta}q_{\alpha\beta})}{dt^3}\right)^2\right\rangle
\end{aligned} \tag{131}$$

Define the reduced quadrupole moment tensor as

$$Q_{\alpha\beta} = q_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}q$$

where $q_{\alpha\beta}$ is the quadrupole moment tensor in (74). Since $Q_{\alpha\beta}$ is traceless by definition, we have

$$P_{\ell m\alpha\beta}Q_{\alpha\beta} = P_{\ell m\alpha\beta}q_{\alpha\beta} \tag{132}$$

Then, we can calculate the GW luminosity L_{GW} from (131) as

$$\begin{aligned}
L_{GW} &\equiv \frac{dE_{GW}}{dt} = \int\frac{dE_{GW}}{dtdS}r^2 d\Omega \\
&= \frac{G}{2c^5}\frac{1}{4\pi}\int d\Omega\left\langle\sum_{\alpha\beta}\left(\frac{d^3(P_{\ell m\alpha\beta}Q_{\alpha\beta})}{dt^3}\right)^2\right\rangle
\end{aligned} \tag{133}$$

By using the definition and properties of $P_{mn\alpha\beta}$ in (82), (83), and (85), we have

$$\begin{aligned}
&\sum_{\alpha\beta}\left(\frac{d^3(P_{\ell m\alpha\beta}Q_{\alpha\beta})}{dt^3}\right)^2 = \sum_{\alpha\beta}\frac{d^3(P_{\ell m\alpha\beta}Q_{\alpha\beta})}{dt^3}\frac{d^3(P_{\ell m\gamma\lambda}Q_{\gamma\lambda})}{dt^3} \\
&= \sum_{\alpha\beta}P_{\alpha\beta\ell m}P_{\ell m\gamma\lambda}\frac{d^3 Q_{\alpha\beta}}{dt^3}\frac{d^3 Q_{\gamma\lambda}}{dt^3} = \sum_{\alpha\beta}P_{\alpha\beta\gamma\lambda}\frac{d^3 Q_{\alpha\beta}}{dt^3}\frac{d^3 Q_{\gamma\lambda}}{dt^3} \\
&= \left[(\delta_{\alpha\gamma} - n_\alpha n_\gamma)(\delta_{\beta\lambda} - n_\beta n_\lambda) - \frac{1}{2}(\delta_{\alpha\beta} - n_\alpha n_\beta)(\delta_{\gamma\lambda} - n_\gamma n_\lambda)\right] \\
&\frac{d^3 Q_{\alpha\beta}}{dt^3}\frac{d^3 Q_{\gamma\lambda}}{dt^3} = \left(\frac{d^3 Q_{\gamma\beta}}{dt^3}\right)^2 - 2n_\alpha n_\gamma\frac{d^3 Q_{\alpha\lambda}}{dt^3}\frac{d^3 Q_{\lambda\gamma}}{dt^3} \\
&\qquad + \frac{1}{2}n_\alpha n_\beta n_\gamma n_\lambda\frac{d^3 Q_{\alpha\beta}}{dt^3}\frac{d^3 Q_{\gamma\lambda}}{dt^3}
\end{aligned}$$

Then, L_{GW} can be calculated by substitution in (133)

$$\frac{dE_{GW}}{dt} = \frac{G}{5c^5}\left\langle\sum_{\alpha,\beta=1}^{3}\frac{d^3 Q_{\alpha\beta}}{dt^3}\frac{d^3 Q_{\alpha\beta}}{dt^3}\right\rangle \tag{134}$$

From (93), the tensor $Q_{\alpha\beta}$ of a binary system can be represented as

$$Q_{\alpha\beta} = \frac{\mu a^2}{2}\begin{bmatrix}\cos 2\omega t & \sin 2\omega t & 0 \\ -\sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0\end{bmatrix}$$

Thus, the third time-derivative of $Q_{\alpha\beta}$ is

$$\frac{d^3 Q_{\alpha\beta}}{dt^3} = \frac{\mu a^2}{2}8\omega^3\begin{bmatrix}\sin 2\omega t & -\cos 2\omega t & 0 \\ -\cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0\end{bmatrix}$$

which implies

$$\sum_{\alpha,\beta=1}^{3} \frac{d^3 Q_{\alpha\beta}}{dt^3} \frac{d^3 Q_{\alpha\beta}}{dt^3} = 32\mu^2 a^4 \omega^6 \tag{135}$$

Finally, by substituting (135) into (134), the rate at which energy is carried away by GWs for a binary system is derived as

$$\frac{dE_{\mathrm{GW}}}{dt} = \frac{32G}{5c^5} \mu^2 a^4 \omega^6$$

which is the equation in (99).