Gravitational Wave Theories

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I. Basic Theories

A. Minkowski Space

An infinitesimal physical displacement $d\bar{s}$ in a four-dimensional space-time can be represented as

$$(ds)^2 = \sum_{\mu=1}^{4}\sum_{\nu=1}^{4} g_{\mu\nu} dx^\mu dx^\nu$$

where

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1}$$

is the metric in the Minkowski space.

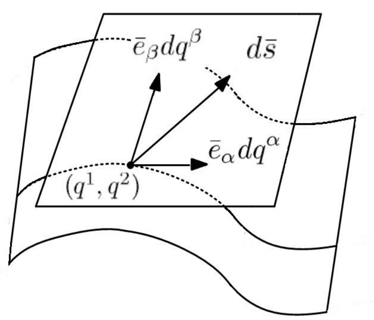

Fig. 1. Tangent plane to a two-dimensional surface of arbitrary shape at the position of the particle (q^1, q^2).

Fig.1 shows a tangent plane of a point particle moving without friction on a two-dimensional surface of arbitrary shape. The generalized coordinates of the particle are (q^1, q^2), and $d\bar{s}$ represents an infinitesimal displacement on the surface, which can be written as

$$d\bar{s} = \sum_{\alpha=1}^{2} \bar{e}_\alpha dq^\alpha \tag{2}$$

From (2), define an n-dimensional infinitesimal displacement as

$$d\bar{s} = \sum_{\alpha=1}^{n} \bar{e}_\alpha dq^\alpha \tag{3}$$

where $(q^1, \cdots q^n)$ are the generalized coordinates and $\bar{e}_\alpha$ is the directional vector tangent to the q_α, which is not normalized. A reciprocal basis $\{\bar{e}^\beta\}$ is defined, which satisfies the orthonormality condition that

$$\bar{e}_\alpha \cdot \bar{e}^\beta = \begin{cases} 1, & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \tag{4}$$

The infinitesimal displacement $d\bar{s}$ can be represented as

$$d\bar{s} = \sum_{\alpha=1}^{n} \bar{e}^\alpha dq_\alpha \tag{5}$$

Next, by taking the inner product of $d\bar{s}$ represented in (3) with itself, we have

$$(ds)^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} g_{\alpha\beta} dq^\alpha dq^\beta \tag{6}$$

where

$$g_{\alpha\beta} = \bar{e}_\alpha \cdot \bar{e}_\beta \tag{7}$$

is called the covariant metric. Similarly, by taking the inner product of $d\bar{s}$ represented in (5) with itself, we have

$$(ds)^2 = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} g^{\alpha\beta} dq_\alpha dq_\beta \tag{8}$$

where

$$g^{\alpha\beta} = \bar{e}^\alpha \cdot \bar{e}^\beta \tag{9}$$

is called the contravariant metric.

From here on, the Einstein notation will be used to simplify the derivations. Consider different curvilinear coordinates (q^1, q^2) and (ξ^1, ξ^2), which are related to each other as

$$d\xi^\alpha = a'^\alpha_\beta dq^\beta$$
$$dq^\alpha = a^\alpha_\beta dq^\beta$$

or

$$a'^\alpha_\beta = \frac{\partial \xi^\alpha}{\partial q^\beta}$$
$$a^\alpha_\beta = \frac{\partial q^\alpha}{\partial \xi^\beta}$$

and $d\bar{s}$ can be represented either as $\bar{e}_\beta dq^\beta$ or $\bar{\varepsilon}_\beta d\xi^\beta$, leading to

$$\bar{e}_\beta dq^\beta = \bar{\varepsilon}_\beta d\xi^\beta \tag{10}$$

where Einstein notation has been used.

By taking the inner product of $\bar{e}^\alpha$ with (10), we obtain

$$a^\alpha_\beta = \bar{e}^\alpha \cdot \bar{\varepsilon}_\beta$$

By taking the inner product of $\bar{\varepsilon}^\alpha$ with (10), we obtain

$$a'^\alpha_\beta = \bar{\varepsilon}^\alpha \cdot \bar{e}_\beta$$

Thus $\bar{\varepsilon}_\alpha$ and $\bar{e}_\alpha$ can be derived as

$$\bar{\varepsilon}_\alpha = (\bar{\varepsilon}_\alpha \cdot \bar{e}^\beta)\bar{e}_\beta = a^\beta_\alpha \bar{e}_\beta \quad (11)$$

$$\bar{e}_\alpha = (\bar{e}_\alpha \cdot \bar{\varepsilon}^\beta)\bar{\varepsilon}_\beta = a'^\beta_\alpha \bar{\varepsilon}_\beta \quad (12)$$

Any vector can be represented as

$$\bar{v} = v^\alpha \bar{e}_\alpha = v_\beta \bar{e}^\beta \quad (13)$$

The notion can be generalized to represent a second-rank tensor as

$$\bar{\bar{T}} = T^{\alpha\beta}\bar{e}_\alpha \bar{e}_\beta = T_{\alpha\beta}\bar{e}^\alpha \bar{e}^\beta \quad (14)$$

where

$$T_{\alpha\beta} = g_{\alpha\alpha'} g_{\beta\beta'} T^{\alpha'\beta'}$$

Similar to (13), a vector can be represented in two different coordinates as

$$\bar{v} = v^\alpha \bar{e}_\alpha = v'^\beta \bar{\varepsilon}_\beta \quad (15)$$

By substituting (11) into (15), we have

$$\bar{v} = v^\alpha \bar{e}_\alpha = v'^\beta a^\alpha_\beta \bar{e}_\alpha \quad (16)$$

leading to

$$v^\alpha = a^\alpha_\beta v'^\beta \quad (17)$$

Similarly, by interchanging the indices in (15), we have

$$\bar{v} = v'^\alpha \bar{\varepsilon}_\alpha = v^\beta \bar{e}_\beta \quad (18)$$

By substituting the relation (12) into (18), we have

$$\bar{v} = v'^\alpha \bar{\varepsilon}_\alpha = v^\beta a'^\alpha_\beta \bar{\varepsilon}_\alpha \quad (19)$$

leading to

$$v'^\alpha = a'^\alpha_\beta v^\beta \quad (20)$$

Any second-rank tensor as (14) can be represented in two different coordinates as

$$\bar{\bar{T}} = T^{\alpha\beta}\bar{e}_\alpha \bar{e}_\beta = T'^{\ell m}\bar{\varepsilon}_\ell \bar{\varepsilon}_m \quad (21)$$

By substituting the relation in (11) into (21), we have

$$\bar{\bar{T}} = T^{\alpha\beta}\bar{e}_\alpha \bar{e}_\beta = T'^{\ell m} a^\alpha_\ell \bar{e}_\alpha a^\beta_m \bar{e}_\beta \quad (22)$$

leading to

$$T^{\alpha\beta} = a^\alpha_\ell a^\beta_m T'^{\ell m} \quad (23)$$

Similarly, by interchanging the indices in (21), we have

$$\bar{\bar{T}} = T'^{\alpha\beta}\bar{\varepsilon}_\alpha \bar{\varepsilon}_\beta = T^{\ell m}\bar{e}_\ell \bar{e}_m \quad (24)$$

By substituting the relation in (12) into (24), we have

$$\bar{\bar{T}} = T'^{\alpha\beta}\bar{\varepsilon}_\alpha \bar{\varepsilon}_\beta = T^{\ell m} a'^\alpha_\ell \bar{\varepsilon}_\alpha a'^\beta_m \bar{\varepsilon}_\beta \quad (25)$$

leading to

$$T'^{\alpha\beta} = a'^\alpha_\ell a'^\beta_m T^{\ell m} \quad (26)$$

Consider the two representations of $d\bar{s}$ in n-dimensional space, as in (3) and (5),

$$d\bar{s} = \bar{e}_\alpha dq^\alpha = \bar{e}^\beta dq_\beta \quad (27)$$

Take the inner product of (27) to $\bar{e}^\alpha$ and $\bar{e}_\beta$, respectively, and use the definitions in (4), (7) and (9) to derive

$$dq^\alpha = g^{\alpha\beta} dq_\beta \quad (28)$$

$$dq_\beta = g_{\alpha\beta} dq^\alpha \quad (29)$$

By substituting (29) into (28), we have

$$dq^\alpha = g^{\alpha\ell} g_{\ell\beta} dq^\beta$$

which implies

$$\left(g^{\alpha\ell} g_{\ell\beta} - \delta^\alpha_\beta\right) dq^\beta = 0$$

Since dq^βs are linearly independent, implying that

$$g^{\alpha\ell} g_{\ell\beta} = \delta^\alpha_\beta \quad (30)$$

B. Particle Motion

Einstein's theory of general relativity claims that the presence of mass can produce a curved four-dimensional space-time such that the geodesics in this space-time reproduce Newton's second law and law of gravitation [1]

$$\frac{d\bar{v}}{dt} = -\frac{GM\bar{r}}{r^3} \quad (31)$$

in an appropriate limit. The particle motion involves only kinetic energy, and no forces are exerted on the particle. Thus, the Hamilton's principle can be applied, leading to the Lagrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial\left(dq^\ell/dt\right)} - \frac{\partial L}{\partial q^\ell} = 0, \quad \ell = 1, 2 \quad (32)$$

By substituting the lagrangian $L = mv^2/2$ into (32), the Lagrange's equation in the generalized coordinates (q^1, q^2) is derived as

$$g_{\ell\beta}\frac{d^2 q^\beta}{dt^2} + \frac{1}{2}\left(\frac{\partial g_{\ell\gamma}}{\partial q^\beta} + \frac{\partial g_{\ell\beta}}{\partial q^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial q^\ell}\right)\frac{dq^\beta}{dt}\frac{dq^\gamma}{dt} = 0 \quad (33)$$

In geodesic motion, the minimum distance between two points satisfies the Lagrange's equation, and the path of the particle is called a geodesic.

By multiplying $g^{\alpha\ell}$ to (33), and imposing (30), we have

$$\frac{d^2 q^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}(q)\frac{dq^\beta}{dt}\frac{dq^\gamma}{dt} = 0 \quad (34)$$

where

$$\Gamma^\alpha_{\beta\gamma}(q) = \frac{1}{2}g^{\alpha\ell}(q)\left[\frac{\partial g_{\ell\gamma}(q)}{\partial q^\beta} + \frac{\partial g_{\ell\beta}(q)}{\partial q^\gamma} - \frac{\partial g_{\beta\gamma}(q)}{\partial q^\ell}\right] \quad (35)$$

Fig.2 shows the generalized coordinates and corresponding basis vectors in a two-dimensional curved space. The change in basis vector $d\bar{e}_\alpha$ can be characterized with the original basis vectors as

$$d\bar{e}_\alpha = X^\gamma_{\alpha\beta}\bar{e}_\gamma dq^\beta \quad (36)$$

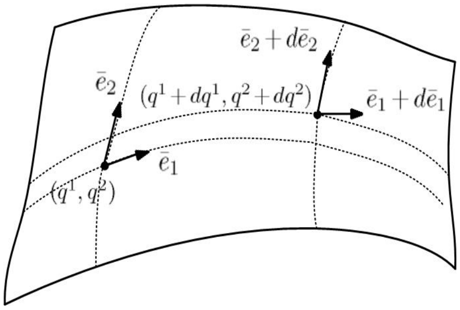

Fig. 2. Changes of basis vectors from point(q^1, q^2) to point ($q^1 + dq^1$, $q^2 + dq^2$) in a curved space.

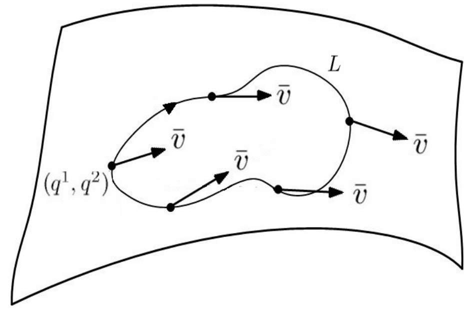

Fig. 3. Parallel transport of a vector $\bar{v}$ around a closed curve L in a two-dimensional curved space, starting from (q^1, q^2).

By (3), we have

$$\bar{e}_\alpha = \frac{\partial \bar{s}}{\partial q^\alpha}$$

It follows that

$$\frac{\partial \bar{e}_\alpha}{\partial q^\beta} = \frac{\partial^2 \bar{s}}{\partial q^\beta \partial q^\alpha} = \frac{\partial^2 \bar{s}}{\partial q^\alpha \partial q^\beta} = \frac{\partial \bar{e}_\beta}{\partial q^\alpha} \tag{37}$$

By substituting (36) into (37), we have

$$X^{\gamma}_{\alpha\beta}\bar{e}_\gamma = X^{\gamma}_{\beta\alpha}\bar{e}_\gamma \tag{38}$$

which means the coefficients $X^{\gamma}_{\alpha\beta}$ are symmetric in the lower two indices.

Next, by taking the differential of (7) and applying (36), the change in metric is derived as

$$\begin{aligned} dg_{\alpha\beta} &= \bar{e}_\alpha \cdot d\bar{e}_\beta + d\bar{e}_\alpha \cdot \bar{e}_\beta \\ &= \bar{e}_\alpha \cdot X^{\gamma}_{\beta\ell}\bar{e}_\gamma dq^\ell + \bar{e}_\beta \cdot X^{\gamma}_{\alpha\ell}\bar{e}_\gamma dq^\ell \\ &= \left(X^{\gamma}_{\beta\ell} g_{\alpha\gamma} + X^{\gamma}_{\alpha\ell} g_{\alpha\gamma} g_{\beta\gamma} \right) dq^\ell \end{aligned} \tag{39}$$

Since the total differential of $g_{\alpha\beta}$ can be represented in the q^ℓ coordinates as

$$g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial q^\ell} dq^\ell \tag{40}$$

By comparing (39) and (40), we have

$$\frac{\partial g_{\alpha\beta}}{\partial q^\ell} = X^{\gamma}_{\beta\ell} g_{\alpha\gamma} + X^{\gamma}_{\alpha\ell} g_{\alpha\gamma} g_{\beta\gamma} \tag{41}$$

Due to the symmetric charateristic in (38), we find that

$$\frac{\partial g_{\ell\gamma}(q)}{\partial q^\beta} + \frac{\partial g_{\ell\beta}(q)}{\partial q^\gamma} - \frac{\partial g_{\beta\gamma}(q)}{\partial q^\ell} = 2X^{\alpha}_{\beta\gamma} g_{\ell\alpha} \tag{42}$$

By multiplying $g_{\alpha\ell}/2$ to (42), then using the relations in (30) and (35), we have

$$\begin{aligned} X^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\ell}(q) \left[\frac{\partial g_{\ell\gamma}(q)}{\partial q^\beta} + \frac{\partial g_{\ell\beta}(q)}{\partial q^\gamma} - \frac{\partial g_{\beta\gamma}(q)}{\partial q^\ell} \right] \\ &= \Gamma^{\alpha}_{\beta\gamma} \end{aligned}$$

Thus, (36) can be rewritten as

$$d\bar{e}_\alpha = \Gamma^{\gamma}_{\alpha\beta} \bar{e}_\gamma dq^\beta \tag{43}$$

II. Einstein Field Equation

A. Riemann Curvature Tensor

Fig.3 shows parallel transport of a vector $\bar{v}$ around a closed curve L in a two-dimensional curved space, where the vector components in the tangent plane are kept constant at all points along the curve. Thus, the vector $\bar{v}$, represented in (13), satisfies

$$d\bar{v} = dv^\alpha \bar{e}_\alpha + v^\alpha d\bar{e}_\alpha = 0 \tag{44}$$

The generalized coordinates q^α along L as

$$q^\alpha = q^\alpha_0 + \epsilon f^\alpha(\tau) \tag{45}$$

$$f^\alpha(1) = f^\alpha(0) \tag{46}$$

where ϵ is a small parameter and $f^\alpha(\tau)$ is a scalar function defined along the closed curve L.

Substituting (43) into (44), we have

$$d\bar{v} = \left(dv^\gamma + v^\alpha \Gamma^{\gamma}_{\alpha\beta} dq^\beta \right) \bar{e}_\gamma = 0$$

which implies

$$dv^\gamma = -v^\alpha \Gamma^{\gamma}_{\alpha\beta} dq^\beta \tag{47}$$

By using (45), (47) is reduced to

$$\frac{dv^\alpha(\tau)}{d\tau} = -\Gamma^{\alpha}_{\beta\gamma}(q) v^\beta(\tau) \epsilon \frac{df^\gamma(\tau)}{d\tau} \tag{48}$$

Next, apply power series expansion of $\Gamma^{\alpha}_{\beta\gamma}$ and $\bar{v}$ along L, with the relation in (45), to have

$$\begin{aligned} \Gamma^{\alpha}_{\beta\gamma}(q) &\simeq \Gamma^{\alpha}_{\beta\gamma}(q_0) + \epsilon f^\ell(\tau) \frac{\partial \Gamma^{\alpha}_{\beta\gamma}(q_0)}{\partial q^\ell} \\ v^\alpha(\tau) &\simeq v^\alpha_0 + \epsilon v^\alpha_1(\tau) + \epsilon^2 v^\alpha_2(\tau) \end{aligned}$$

which are then substituted into (48) to obtain

$$\begin{aligned} &\epsilon \frac{dv^\alpha_1(\tau)}{d\tau} + \epsilon^2 \frac{dv^\alpha_2(\tau)}{d\tau} \\ &\simeq -\epsilon \frac{df^\gamma(\tau)}{d\tau} \left[\Gamma^{\alpha}_{\beta\gamma}(0) + \epsilon f^\ell(\tau) \frac{\partial \Gamma^{\alpha}_{\beta\gamma}(0)}{\partial q^\ell} \right] \left[v^\beta_0 + \epsilon v^\beta_1(\tau) \right] \end{aligned}$$

where $\Gamma^{\alpha}_{\beta\gamma}(0) = \Gamma^{\alpha}_{\beta\gamma}(\tau = 0) = \Gamma^{\alpha}_{\beta\gamma}(q_0)$. By equating the coefficients in the first-order (ϵ) terms, we have

$$v^\alpha_1(\tau) = -\Gamma^{\alpha}_{\beta\gamma}(0) v^\beta_0 f^\gamma(\tau) \tag{49}$$

Similarly, by equating the coefficients in the second-order (ϵ^2) terms, we find

$$\begin{aligned}\frac{dv_2^\alpha(\tau)}{d\tau} &= -\left[\frac{\partial\Gamma^\alpha_{\beta\gamma}(0)}{\partial q^\ell} f^\ell(\tau)\frac{df^\gamma(\tau)}{d\tau}v_0^\beta \right.\\ &\left. +\Gamma^\alpha_{\beta\gamma}(0)v_1^\beta(\tau)\frac{df^\gamma(\tau)}{d\tau}\right]\\ &= -\left[\frac{\partial\Gamma^\alpha_{\beta\gamma}(0)}{\partial q^\ell} - \Gamma^\alpha_{m\gamma}(0)\Gamma^m_{\beta\ell}(0)\right]v_0^\beta f^\ell(\tau)\frac{df^\gamma(\tau)}{d\tau} \quad (50)\end{aligned}$$

Before integrating (50) around L, we first perform the following integration

$$\begin{aligned}\oint_L f^\ell \frac{df^\gamma}{d\tau}d\tau &= \frac{1}{2}\oint_L\left[\left(f^\ell\frac{df^\gamma}{d\tau} - f^\gamma\frac{df^\ell}{d\tau}\right)\right.\\ &\left.+\frac{d}{d\tau}(f^\ell f^\gamma)\right]d\tau = \frac{1}{2}\oint_L\left(f^\ell\frac{df^\gamma}{d\tau} - f^\gamma\frac{df^\ell}{d\tau}\right)\\ &= \frac{1}{2\epsilon^2}\oint_L\left[(q^\ell - q_0^\ell)\frac{dq^\gamma}{d\tau} - (q^\gamma - q_0^\gamma)\frac{dq^\ell}{d\tau}\right]d\tau = \frac{1}{\epsilon^2}S^{\ell\gamma} \quad (51)\end{aligned}$$

where

$$S^{\ell\gamma} = \frac{1}{2}\oint_L (q^\ell dq^\gamma - q^\gamma dq^\ell) = -S^{\gamma\ell} \quad (52)$$

Define a Riemann curvature tensor as

$$\begin{aligned}R^\alpha_{\beta\ell\gamma} &= \left[\frac{\partial\Gamma^\alpha_{\beta\gamma}(0)}{\partial q^\ell} - \Gamma^\alpha_{m\gamma}(0)\Gamma^m_{\beta\ell}\right]\\ &- \left[\frac{\partial\Gamma^\alpha_{\beta\ell}(0)}{\partial q^\gamma} - \Gamma^\alpha_{m\ell}(0)\Gamma^m_{\beta\gamma}\right]\\ &= \left(\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial q^\ell} + \Gamma^\alpha_{m\ell}\Gamma^m_{\beta\gamma}\right) - \left(\frac{\partial\Gamma^\alpha_{\beta\ell}}{\partial q^\gamma} + \Gamma^\alpha_{m\gamma}\Gamma^m_{\beta\ell}\right) \quad (53)\end{aligned}$$

where the argument (0) is omitted because the reference point q_0 can be chosen arbitarily.

By using the antisymmetry of $S^{\ell\gamma}$ in (52) and $R^\alpha_{\beta\ell\gamma}$ in (53) with respect to indices γ and ℓ, we have

$$R^\alpha_{\beta\ell\gamma}S^{\ell\gamma} = \frac{1}{2}(R^\alpha_{\beta\ell\gamma}S^{\ell\gamma} + R^\alpha_{\beta\gamma\ell}S^{\gamma\ell})$$

Finally, by integrating (50) over the closed loop L, we obtain a non-zero second-order change in $\bar{v}$ as

$$\Delta v^\alpha = v_2^\alpha(1) - v_2^\alpha(0) = -\frac{1}{2\epsilon^2}R^\alpha_{\beta\ell\gamma}v_0^\beta S^{\ell\gamma}$$

B. Energy-Momentum Tensor

The energy-momentum tensor is the source of the gravitational field in the Einstein field equation, just as mass density is the source of such a field in Newtonian gravity.

In four-dimensional space-time, a four-vector can be represented as in (13). Eqn.(6) is repeated as

$$(d\bar{s})^2 = g_{\mu\nu}dx^\mu dx^\nu = d\bar{x}\cdot d\bar{x} - c^2(dt)^2 = -c^2(d\tau)^2 \quad (54)$$

where $dx^\mu = dq^\mu$ and τ is called the proper time. Dividing (54) by $(dt)^2$ leads to

$$\left(\frac{d\bar{s}}{dt}\right)^2 = \left(\frac{d\bar{x}}{dt}\right)^2 - c^2 = \bar{v}^2 - c^2 = -c^2\left(\frac{d\tau}{dt}\right)^2$$

Thus, time and proper time are related as

$$\frac{d\tau}{dt} = \sqrt{1-\beta^2} \quad (55)$$

with

$$\bar{\beta} = \frac{\bar{v}}{c}$$

Define the four-velocity as

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}}[\bar{v}, c] \quad (56)$$

$$u_\mu = g_{\mu\nu}u^\nu = \frac{1}{\sqrt{1-\beta^2}}[\bar{v}, -c] \quad (57)$$

where (55) is used.

Consider an inertial frame F' which moves at a relative velocity $\bar{v}$ with respect to F. Thus, the origin of F' is located at a position $\bar{x}_0$ in frame F. If the clocks are reset at the first event where two origins coincide, and the second event happens at the origin of F', we can write

$$\bar{x} = \bar{v}t + ct\bar{e}_4 = \bar{0} + ct'\bar{\varepsilon}_4$$

where $\bar{x}$ is the vector connecting the two events in space-time, leading to

$$\bar{\varepsilon}_4 = \frac{t}{t'}(\bar{\beta} + \bar{e}_4) \quad (58)$$

The proper time is the time measured by an observer who is moving with the frame. Thus $t' = \tau$, and (55) is reduced to

$$\bar{\varepsilon}_4 = \frac{\bar{\beta} + \bar{e}_4}{\sqrt{1-\beta^2}} \quad (59)$$

If $\bar{e}_3$ is along the direction of velocity, (59) becomes

$$\bar{\varepsilon}_4 = \frac{\beta\bar{e}_3 + \bar{e}_4}{\sqrt{1-\beta^2}}$$

Next, by imposing that $\bar{\varepsilon}_3\cdot\bar{\varepsilon}_4 = 0$ and $\bar{\varepsilon}_3\cdot\bar{\varepsilon}_3 = 1$, we derive

$$\bar{\varepsilon}_3 = \frac{\bar{e}_3 + \beta\bar{e}_4}{\sqrt{1-\beta^2}}$$

Thus, by using the relation in (11), the Lorentz transformation matrix is obtained as

$$a^\mu_\nu = \frac{1}{\sqrt{1-\beta^2}}\begin{bmatrix}\sqrt{1-\beta^2} & 0 & 0 & 0\\ 0 & \sqrt{1-\beta^2} & 0 & 0\\ 0 & 0 & 1 & \beta\\ 0 & 0 & \beta & 1\end{bmatrix} \quad (60)$$

which is also labeled as

$$\underline{a}_{\mu\nu} = a^\mu_\nu$$

By comparing (56) and (60), it is observed that

$$\underline{a}_{\mu 4} = \frac{1}{c}u^\mu = \underline{a}_{4\mu} = \frac{1}{\sqrt{1-\beta^2}}[\bar{\beta}, 1] \quad (61)$$

C. Isotropic Fluid with No Shear Forces

In special relativity, a second-rank energy-momentum tensor exists for an isotropic fluid with no shear forces, which can be represented as

$$\bar{\bar{T}} = T^{\mu\nu}\bar{e}_\mu\bar{e}_\nu = T'^{\mu\nu}\bar{\varepsilon_\mu}\bar{\varepsilon_\nu}$$

where

$$T^{\mu\nu} = a^\mu_{\mu'}a^\nu_{\nu'}T'^{\mu'\nu'} \tag{62}$$

by applying (21) and (23), and $T'^{\mu\nu}$ in the rest frame of fluid can be represented as

$$T'^{\alpha\beta} = P\delta^{\alpha\beta} \tag{63}$$

$$T'^{44} = \rho c^2 \tag{64}$$

where $T'^{\alpha\beta}$ in (63) indicates the spatial components of pressure, and T'^{44} in (64) is the proper energy density.

By substituting (63) and (64) into (62), we have

$$T^{\mu\nu} = Pa^\mu_m a^\nu_m + \rho c^2 a^\mu_4 a^\nu_4 \tag{65}$$

From (60) and (61), we find

$$a^\mu_m a^\nu_m = g^{\mu\nu} + \frac{1}{c^2}u^\mu u^\nu$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+\beta^2/(1-\beta^2) & \beta/(1-\beta^2) \\ 0 & 0 & \beta/(1-\beta^2) & -1+1/(1-\beta^2) \end{bmatrix}$$
$$a^\mu_4 a^\nu_4 = \frac{1}{c^2}u^\mu u^\nu$$

Thus, the energy-momentum tensor in (65) can be reduced to

$$T^{\mu\nu} = Pg^{\mu\nu} + \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu \tag{66}$$

D. Field Equations

The Einstein tensor $G^{\mu\nu}$ is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{67}$$

where

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{68}$$

is called the Ricci tensor, and

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu}$$

is called the scalar curvature.

Similarly, a scalar can be defined from any tensor as

$$T = T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} \tag{69}$$

Assume that the structure of space-time satisfies a field equation

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{70}$$

where κ is a constant to be determined. The tensors are both symmetric and satisfy the energy-momentum conservation

$$\nabla^\mu T_{\mu\nu} = 0$$

Thus, we only have to check if it actually produces gravity as we know it.

By multiplying $g^{\mu\nu}$ to the left-hand side of (70) and using (67), we have

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})$$
$$= R - 2R = -R$$

By multiplying $g^{\mu\nu}$ to the right-hand side of (70) and using (69), we have

$$\kappa g^{\mu\nu}T_{\mu\nu} = \kappa T$$

Thus, (70) implies that

$$R = -\kappa T \tag{71}$$

By expressing $G_{\mu\nu}$ with (67) and replacing R with $-\kappa T$ as in (71), the Einstein equation in (70) can be represented in another form as

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) \tag{72}$$

The gravitational field, following (31), is represented as

$$\bar{g} = -\hat{r}\frac{GM}{r^2} \tag{73}$$

Because gravitational field is conservative, it can be represented in terms of a gravitational potential field Φ as

$$\bar{g} = -\nabla\Phi \tag{74}$$

In the special case of a point mass, Φ can be expressed as

$$\Phi = -\frac{GM}{r} \tag{75}$$

Next, apply the Gauss's law to (73) over a sphere S of radius r to have

$$\oint_S \bar{g}\cdot d\bar{a} = -4\pi GM \tag{76}$$

where $d\bar{a}$ is an infinitesimal surface on S. By applying the divergence theorem, the left-hand side of (76) is reduced to

$$\oint_S \bar{g}\cdot d\bar{S} = \int_V \nabla\cdot\bar{g}dv \tag{77}$$

The total mass M is the volume integral of mass density ρ as

$$M = \int_V \rho dv \tag{78}$$

By substituting (77) and (78) into (76), we have

$$\nabla\cdot\bar{g} = -4\pi G\rho \tag{79}$$

Then, by substituting (74) into (79), we have

$$\nabla^2\Phi = 4\pi G\rho \tag{80}$$

In the weak-field limit, a perturbation $h_{\mu\nu}(\bar{x})$ is superimposed upon the flat Minkowski metric $g^0_{\mu\nu}$ in (1) as

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}(\bar{x}) \tag{81}$$

A particle follows the geodesic expressed in (34), namely,

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{44}c^2 = 0 \quad (82)$$

where only the coordinate $x^4 = ct$ contributes to the second term in (34).

Next, by substituting (81) into the affine connection in (35), and assuming $h_{\mu\nu}(\bar{x})$ is independent of time, we have

$$\begin{aligned}\Gamma^\mu_{44} &\simeq \frac{1}{2}[g^0]^{\mu\sigma}\left[\frac{\partial h_{\sigma 4}(\bar{x})}{\partial ct} + \frac{\partial h_{\sigma 4}(\bar{x})}{\partial ct} - \frac{\partial h_{44}(\bar{x})}{\partial x^\sigma}\right] \\ &= -\frac{1}{2}[g^0]^{\mu\sigma}\left[\frac{\partial h_{44}(\bar{x})}{\partial x^\sigma}\right]\end{aligned} \quad (83)$$

where $\mu, \sigma = 1, 2, 3$.

By substituting (83) into (82) and imposing the Newton's law in (74), we have

$$\frac{d^2\bar{x}}{dt^2} = -\bar{\nabla}\Phi = \nabla\left[\frac{c^2}{2}h_{44}(\bar{x})\right]$$

which implies

$$h_{44}(\bar{x}) = -\frac{2\Phi}{c^2} \quad (84)$$

From (81), we have

$$g_{44} = -1 + h_{44} \quad (85)$$

In the Newtonian limit, the rest energy $T_{44} = \rho c^2$ is the dominant term in $T_{\mu\nu}$. Thus, neglecting the other terms in (69), we have

$$T = g^{44}T_{44} \simeq -T_{44} \quad (86)$$

by neglecting the small perturbation h_{44}.

Substituting (86) into (72), we have

$$R_{44} = \frac{1}{2}\kappa T_{44} \quad (87)$$

Then, R_{44} can be derived from the definition of Ricci tensor in (68), which involves (53), as

$$\begin{aligned}R_{44} &= \left(\frac{\partial\Gamma^\lambda_{44}}{\partial x^\lambda} + \Gamma^\lambda_{m\lambda}\Gamma^m_{44}\right) - \left(\frac{\partial\Gamma^\lambda_{4\lambda}}{\partial x^4} + \Gamma^\lambda_{m4}\Gamma^m_{4\lambda}\right) \\ &= \frac{\partial\Gamma^\lambda_{44}}{\partial x^\lambda}\end{aligned} \quad (88)$$

where the squared terms in Γ are neglected and the third term vanishes in a static field

Next, substitute (83) into (88) to have

$$R_{44} = -\frac{1}{2}\nabla^2 h_{44} \quad (89)$$

By comparing (87) and (89), we obtain

$$\nabla^2 h_{44} = -\kappa T_{44} \quad (90)$$

In Newtonian physics, the mass density ρc^2 is the only source of gravity, thus the energy-momentum tensor in (63) and (64) is reduced to

$$\begin{aligned}T_{\alpha\beta} &= 0 \\ T_{44} &= \rho c^2\end{aligned} \quad (91)$$

By substituting (84) into (80), we have

$$4\pi G\rho = -\frac{c^2}{2}\nabla^2 h_{44} \quad (92)$$

Then, substitute (91) and (90) into (92) to have

$$\kappa = \frac{8\pi G}{c^4}$$

Thus, the Einstein's field equation in (70) is reduced to

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu} \quad (93)$$

E. Uniform Universe

If the mass density is uniform throughout the universe, the energy-momentum tensor takes the approximate form

$$T^{\mu\nu} = \rho u^\mu u^\nu$$

because the pressure contribution in (66) is negligible for the entire universe. For a fluid at rest in the lab frame, the four-velocity in (56) and (57) reduces to

$$\begin{aligned}\frac{u^\mu}{c} &= (0, 0, 0, 1) \\ \frac{u_\mu}{c} &= (0, 0, 0, -1) \\ \frac{u^\mu u_\mu}{c^2} &= -1\end{aligned}$$

leading to

$$T = T^\mu_\mu = g^{\mu\nu}T_{\mu\nu} = -\rho c^2$$

where (69) is applied.

Thus, the source term in the Einstein field equation (72) becomes

$$T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} = \rho c^2\left(\frac{1}{2}g_{\mu\nu} + \frac{u_\mu u_\nu}{c^2}\right)$$

III. Gravitational Plane Wave

We start with cartesian coordinates (x^1, x^2, x^3, ct) in a flat Minkowski space and the Lorentz metric of (1). Suppose there is a small distortion of the space so that this metric is changed to

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

where $g^0_{\mu\nu}$ is the Minkowski metric, and $h_{\mu\nu}$ is a small perturbation. The coordinates q^μ and x^μ are assumed to differ by h in first order

$$dq^\mu = dx^\mu + O(h)$$

The Ricci tensor defned in (68) is now represented as

$$\begin{aligned}R_{\mu\nu} &= \frac{\partial}{\partial q^\lambda}\Gamma^\lambda_{\mu\nu} - \frac{\partial}{\partial q^\nu}\Gamma^\lambda_{\mu\lambda} + O(h^2) \\ &= \frac{\partial}{\partial q^\lambda}\left\{\frac{1}{2}g_0^{\lambda\rho}\left[\frac{\partial h_{\rho\mu}}{\partial q^\nu} + \frac{\partial h_{\rho\nu}}{\partial q^\mu} - \frac{\partial h_{\mu\nu}}{\partial q^\rho}\right]\right\} \\ &\quad -\frac{\partial}{\partial q^\nu}\left\{\frac{1}{2}g_0^{\lambda\rho}\left[\frac{\partial h_{\rho\mu}}{\partial q^\lambda} + \frac{\partial h_{\rho\lambda}}{\partial q^\mu} - \frac{\partial h_{\mu\lambda}}{\partial q^\rho}\right]\right\}\end{aligned} \quad (94)$$

Through $O(h)$, the metric g_0 can raise indices on h, thus

$$g_0^{\lambda\rho}h_{\rho\nu} = h^\lambda_\nu \quad (95)$$

Then $R_{\mu\nu}$ in (94) becomes

$$R_{\mu\nu} = -\frac{1}{2}g_0^{\lambda\rho}\frac{\partial^2}{\partial q^\lambda \partial q^\rho}h_{\mu\nu} + \frac{1}{2}\frac{\partial^2}{\partial q^\lambda \partial q^\mu}h_\nu^\lambda + \frac{1}{2}\frac{\partial^2}{\partial q^\nu \partial q^\rho}h_\mu^\rho - \frac{1}{2}\frac{\partial^2}{\partial q^\nu \partial q^\mu}h \tag{96}$$

We define a new tensor $\psi_\mu^{\bullet\nu}$ by

$$\psi_\mu^{\bullet\nu} = h_\mu^{\bullet\nu} - \frac{1}{2}h\delta_\mu^{\bullet\nu} = \psi_\mu^\nu \tag{97}$$

With

$$\frac{\partial}{\partial q^\mu} = \frac{\partial}{\partial x^\mu} + O(h)$$

we find that

$$g_0^{\lambda\rho}\frac{\partial^2}{\partial q^\lambda \partial q^\rho} = g_0^{\lambda\rho}\frac{\partial^2}{\partial x^\lambda \partial x^\rho} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Now, the ricci tensor in (96) can be rewritten as

$$R_{\mu\nu} = -\frac{1}{2}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} + \frac{1}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda}\psi_\nu^{\bullet\lambda} + \frac{1}{2}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\lambda}\psi_\mu^{\bullet\lambda} \tag{98}$$

We could pick a corresponding set of generalized coordinates in the deformed space so that the following auxiliary condition is satisfied

$$\frac{\partial}{\partial x^\lambda}\psi_\nu^{\bullet\lambda} = 0 \tag{99}$$

where ν = 1,2,3,4. The Ricci tensor then takes the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = -2R_{\mu\nu}$$

In free space where $T_{\mu\nu}$ = 0, the Einstein equation in (93) reduced to the form of the wave equation for the metric

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} = 0 \tag{100}$$

Then we can find a solution for gravitational plane wave propagating in the z-direction as

$$h_{mn} = (h_0)_{mn}e^{ik(z-ct)} \tag{101}$$

where $(m, n) = (x, y)$ and $(h_0)_{mn}$ = constant, only the spatial parts of the metric (h_{xx}, h_{xy}, h_{yy}) are deformed, and no modification of the z-coordinate and time t.

IV. Schwarzschild Metric

We now consider the solution of Ricci tensor outside a spherically symmetric mass distibution. Assume that the metric is in the form

$$(ds)^2 = A(dr)^2 - B(cdt)^2 + r^2[(d\theta^2) + sin^2\theta(d\phi)^2] \tag{102}$$

where A and B only depend on r. By the assumption in (102), we first compute all the Γs with indices r, θ, ϕ with (35). Next, we compute all the Ricci tensors with indices r, θ, ϕ by substituting Γs into the definition in (68) and (53). Finally, we have

$$\begin{aligned} R_{\theta\theta} &= 1 - \frac{1}{A} + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \\ R_{rr} &= -\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rA} \\ R_{44} &= \frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rA} \end{aligned} \tag{103}$$

and all other terms are zero.

Assuming $AB = \lambda = \text{const}$, we have

$$\frac{d}{dr}(AB) = A'B + AB' = 0$$

Thus,

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{104}$$

To satisfy $R_{\mu\nu} = 0$, the four terms in (103) should be zero. We substitute (104) into (103), leading to

$$\begin{aligned} R_{\theta\theta} &= 1 - \frac{1}{A} + \frac{rA'}{A^2} = 0 \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta = 0 \\ R_{rr} &= -\frac{B''}{2B} + \frac{A'}{rA} = 0 \\ R_{44} &= \frac{B''}{2A} + \frac{B'}{rA} = 0 \end{aligned} \tag{105}$$

From $R_{\theta\theta}$ in (105), we observe that

$$\frac{1}{A} - \frac{rA'}{A^2} = \frac{d}{dr}\left(\frac{r}{A}\right) = 1$$

Therefore, by applying $A = \frac{\lambda}{B}$, we have

$$\frac{d(rB)}{dr} = \lambda \tag{106}$$

Next, using the relation in (104), R_{rr} in (105) is reduced to

$$-rB'' - 2B' = 0$$

which implies

$$\frac{d(r^2B')}{dr} = 0 \tag{107}$$

For a solution to (106), we take

$$rB = \lambda(r - k)$$

or

$$B = \lambda\left(1 - \frac{k}{r}\right) \tag{108}$$

where k is a constant. Applying (108), we have

$$r^2B' = r^2\left(\frac{\lambda k}{r^2}\right) = \lambda k = const$$

which satisfies Eqn.(107).

We define $k = \frac{2GM}{c^2}$, and rescale the coordinate $t \to t/\sqrt(\lambda)$. By substituting the value of k into (108) and using $A = \frac{\lambda}{B}$, B and A are solved as

$$B = 1 - \frac{2GM}{c^2 r}$$
$$A = \frac{1}{1-(2GM/c^2 r)}$$

Thus, the Schwarzschild metric in (102) can be rewritten as

$$(ds)^2 = \frac{1}{1-(2GM/c^2 r)}(dr)^2 - \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 + r^2[(d\theta^2) + \sin^2\theta (d\phi)^2] \tag{109}$$

Consider the Newnonian limit of the Schwarzschild metric. For $r \to \infty$ and $c \to \infty$, (109) is reduced to

$$(ds)^2 \simeq (d\bar{x})^2 - \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 \tag{110}$$

If we substitute the gravitational potential of the Newtonian gravity in (75) into (84), we have

$$h_{44} = \frac{2GM}{c^2 r}$$

Thus, by (85),

$$g_{44} = -(1 - \frac{2GM}{c^2 r})$$

which reproduces the same result as the Schwarzschild metric in the Newnonian limit in (110).

V. Linear Perturbation Theory

To describe gravitational waves as linear perturbations on a flat background spacetime, the metric components can be approximated as [5]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(\epsilon^2) \tag{111}$$

where $\eta_{\mu\nu}$ represents the Minkowski metric, ϵ is the order of magnitude of $h_{\mu\nu}$, which is much smaller than unity. Consider a planar gravitational wave propagating in vacuum in the z direction. The only components of metric perturbation which might not vanish are $h_{\alpha\beta}$, with $\alpha, \beta = 1, 2$, where α and β are used to index the two-dimensional coordinates transverse to the propagation direction of the gravitational wave. Thus, the 2×2 symmetric matrix $\{h_{\alpha\beta}\}$ depend only on the phase coordinate

$$u = \frac{ct - z}{\sqrt{2}} \tag{112}$$

and the first-order line element can be represented as

$$ds^2 = -(cdt)^2 + [\delta_{\alpha\beta} + h_{\alpha\beta}(u)]dx^\alpha dx^\beta + dz^2 + O(\epsilon^2) \tag{113}$$

Assume that only the (x, y) dimensions of the metric are deformed, and the z and ct dimensions are unaffected. Since the spacetime metric $g_{\mu\nu}$ is symmetric, the perturbational terms $h_{\mu\nu}$ in the metric are assumed to be symmetric, namely,

$$h_{xy} = h_{yx} \tag{114}$$

The Ricci tensor of a linearized gravitational wave takes the form [1]

$$R_{\mu\nu} = -\frac{1}{2}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h_{\mu\nu} + \frac{1}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\lambda}\psi_\nu^{\bullet\lambda} + \frac{1}{2}\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\lambda}\psi_\mu^{\bullet\lambda} \tag{115}$$

where $\psi_\mu^{\bullet\nu}$ is defined as

$$\psi_\mu^{\bullet\nu} = h_\mu^{\bullet\nu} - \frac{1}{2}h\delta_\mu^{\bullet\nu} = \psi_\mu^\nu \tag{116}$$

with $h = h_{xx} + h_{yy}$, and $\psi_\mu^{\bullet\nu}$ satisfies the auxiliary condition

$$\frac{\partial \psi_\mu^{\bullet\lambda}}{\partial x^\lambda} = 0 \tag{117}$$

Eqn.(116) can thus be expressed as

$$\psi_x^{\bullet x} = h_{xx} - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{xx} - h_{yy})$$
$$\psi_x^{\bullet y} = h_{xy}$$
$$\psi_y^{\bullet y} = h_{yy} - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{yy} - h_{xx})$$
$$\psi_z^{\bullet z} = \psi_4^{\bullet 4} = -\frac{1}{2}(h_{xx} + h_{yy})$$

and the terms of $\psi_\mu^{\bullet\nu}$ with other combination of μ and ν are zero. By imposing (117) with $\mu = 1, 2, 3$ and 4, respectively, we have

$$\frac{\partial \psi_x^{\bullet\lambda}}{\partial x^\lambda} = \frac{\partial \psi_x^{\bullet x}}{\partial x} + \frac{\partial \psi_x^{\bullet y}}{\partial y} = 0$$
$$\frac{\partial \psi_y^{\bullet\lambda}}{\partial x^\lambda} = \frac{\partial \psi_y^{\bullet x}}{\partial x} + \frac{\partial \psi_y^{\bullet y}}{\partial y} = 0$$
$$\frac{\partial \psi_z^{\bullet\lambda}}{\partial x^\lambda} = \frac{\partial \psi_z^{\bullet z}}{\partial z} = -\frac{1}{2}\frac{\partial (h_{xx} + h_{yy})}{\partial z} = 0$$
$$\frac{\partial \psi_4^{\bullet\lambda}}{\partial x^\lambda} = \frac{\partial \psi_4^{\bullet 4}}{\partial (ct)} = -\frac{1}{2}\frac{\partial (h_{xx} + h_{yy})}{\partial (ct)} = 0 \tag{118}$$

Since $h_{\alpha\beta}$ depends only on z and ct, it is trivial that the first two equations hold. The last two equations imply that

$$h_{xx} + h_{yy} = 0$$

or

$$h_{xx} = -h_{yy} \tag{119}$$

By imposing (114) and (119), $h_{\alpha\beta}(u)$ in (113) can be represented as

$$h_{\alpha\beta}(u) = \begin{bmatrix} h_+(u) & h_\times(u) \\ h_\times(u) & -h_+(u) \end{bmatrix} \tag{120}$$

where $h_+(u)$ and $h_\times(u)$ characterize two different polarization states, with h_+ called the plus polarization and $h_\times$ the cross polarization.

Under the auxiliary condition (117), the solution of (115) can be represented as a plane wave propagating in the z-direction as

$$h_{\alpha\beta} = h_{\alpha\beta 0} f(u)$$

where $f(\bullet)$ is an arbitrary waveform.

The Rosen coordinates (u, v, x^1, x^2) can also be used to describe non-perturbative optical observables in the presence of a gravitational wave. The line element in Rosen coordinates can be represented as

$$ds^2 = -2du dv + \gamma_{\alpha\beta}(u) dx^\alpha dx^\beta \tag{121}$$

where the relation with quasi-Cartesian coordinates is

$$\begin{aligned} ct &= \frac{1}{\sqrt{2}}(v+u) \\ z &= \frac{1}{\sqrt{2}}(v-u) \end{aligned} \tag{122}$$

which implies that $v = (ct+z)/\sqrt{2}$. By substituting (122) into (121), we obtain the exact line element

$$ds^2 = -(cdt)^2 + \gamma_{\alpha\beta}(u) dx^\alpha dx^\beta + dz^2 \tag{123}$$

By equating (123) to (113), we have

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta} + O(\epsilon^2)$$

which takes the form of (111).

VI. Detection of Gravitational Waves

By substituting the matrix $h_{\alpha\beta}$ in (120) into (113), we have

$$\begin{aligned} ds^2 = &-(cdt)^2 + [1 + h_+(u)]dx^2 + [1 - h_+(u)]dy^2 \\ &+2h_\times(u) dx dy + dz^2 \end{aligned} \tag{124}$$

Thus, a general gravitational plane wave propagating in the z-direction can be represented as

$$h_{\mu\nu}(t,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(u) & h_\times(u) & 0 \\ 0 & h_\times(u) & -h_+(u) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{125}$$

Consider two test masses placed at points A and B, respectively. The initial coordinates of points A and B are $\bar{x}^i_A = (ct, 0, 0, 0)$ and $\bar{x}^i_B = (ct, x_B, y_B, z_B)$, respectively. Consider a plus-polarized gravitational plane wave propagting in the z-direction, which can be represented as

$$h_{\mu\nu}(t,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(u) & 0 & 0 \\ 0 & 0 & -h_+(u) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{126}$$

with the corresponding line element

$$ds^2 = -(cdt)^2 + (1 + h_{xx})dx^2 + (1 - h_{xx})dy^2 + dz^2$$

The proper distance between points A and B at fixed (z, ct) can thus be calculated as

$$\Delta\ell = \sqrt{\Delta s^2} = \sqrt{(1 + h_{xx})\Delta x^2 + (1 - h_{xx})\Delta y^2} \tag{127}$$

If both points A and B lie on the x-axis, (127) at $z = 0$ becomes

$$\Delta\ell_x = \sqrt{1 + h_{xx}(t,0)}\Delta x \simeq \left[1 + \frac{1}{2}h_{xx}(t,0)\right]\Delta x \tag{128}$$

which implies the change in distance between the two mirrors is

$$\frac{\Delta\ell_x - \Delta x}{\Delta x} = \frac{1}{2}h_{xx}(t,0) = \frac{1}{2}h_+(ct)$$

which reveals the temporal waveform of the gravitational wave.

Similarly, if both points A and B lie on the y-axis, (127) becomes

$$\Delta\ell_y = \sqrt{1 - h_{xx}(t,0)}\Delta y \simeq \left[1 - \frac{1}{2}h_{xx}(t,0)\right]\Delta y \tag{129}$$

Therefore, the change in distance between the two test masses is

$$\frac{\Delta\ell_y - \Delta y}{\Delta y} = -\frac{1}{2}h_{xx}(t,0) = -\frac{1}{2}h_+(ct)$$

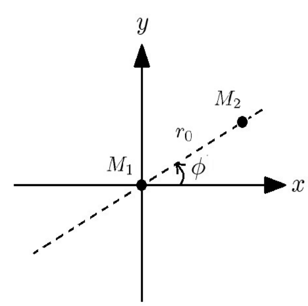

Fig. 4. A test mass M_2 with a distance r_0 from the test mass M_1 at the origin and an angle ϕ from the x-axis.

Fig.4 shows a test mass M_2 with a distance r_0 from the test mass M_1 at the origin and an angle ϕ from the x-axis. Thus, the relative coordinates of M_2 with respect to M_1, can be represented by using (128) and (129) as

$$\begin{aligned} x(t) &= \left[1 + \frac{1}{2}h_{xx}(t,0)\right] r_0 \cos\phi \\ y(t) &= \left[1 - \frac{1}{2}h_{xx}(t,0)\right] r_0 \sin\phi \end{aligned} \tag{130}$$

By eliminating the $h_{xx}(t,0)$ terms in (130), we derive

$$\frac{x}{r_0\cos\phi} + \frac{y}{r_0\sin\phi} = 2$$

which implies that the test mass M_2 moves around its initial position along a straight line with a slope of $-\tan\phi$.

Consider multiple test masses forming a perfect circle with radius r_0 on the xy-plane. By eliminating the ϕ terms in (130), we obtain

$$\frac{x^2}{[a_+(t)]^2} + \frac{y^2}{[b_+(t)]^2} = 1 \tag{131}$$

which is an ellipse with semi-axes

$$a_+(t) = \left[1 + \frac{1}{2}h_{xx}(t,0)\right] r_0$$
$$b_+(t) = \left[1 - \frac{1}{2}h_{xx}(t,0)\right] r_0$$

When $h_{xx}(t,0) > 0$, the circle of test masses will be stretched in the x-direction and squeezed in the y-direction. When $h_{xx}(t,0) < 0$, the circle will be stretched in the y-direction and squeezed in the x-direction.

Next, consider a cross-polarization gravitational plane-wave propagting in the z direction as

$$h_{\mu\nu}(t,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\times(u) & 0 \\ 0 & h_\times(u) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{132}$$

with the corresponding line element

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + 2h_{xy}dxdy + dz^2$$

The proper distance $d\ell$ between two test masses at a given u can be represented as

$$\Delta\ell = \sqrt{ds^2} = \sqrt{\Delta x^2 + \Delta y^2 + 2h_{xy}\Delta x \Delta y} \tag{133}$$

which implies that the distance change in x direction is

$$\Delta\ell_x = \sqrt{\Delta x^2 + h_{yx}\Delta y \Delta x} = \Delta x \sqrt{1 + h_{xy}\frac{\Delta y}{\Delta x}}$$
$$\simeq \Delta x\left(1 + \frac{1}{2}h_{xy}\frac{\Delta y}{\Delta x}\right) = \Delta x + \frac{1}{2}h_{xy}\Delta y \tag{134}$$

and the distance change in the y direction is

$$\Delta\ell_y = \sqrt{\Delta y^2 + h_{xy}\Delta x \Delta y} = \Delta y \sqrt{1 + h_{xy}\frac{\Delta x}{\Delta y}}$$
$$\simeq \Delta y\left(1 + \frac{1}{2}h_{xy}\frac{\Delta x}{\Delta y}\right) = \Delta y + \frac{1}{2}h_{xy}\Delta x \tag{135}$$

If a test mass is placed with a distance r_0 from the origin and an angle ϕ from the x axis, then its coordinates can be determined by using (134) and (135) as

$$x(t) = r_0\left[\cos\phi + \frac{1}{2}\sin\phi h_{xy}(t,0)\right]$$
$$y(t) = r_0\left[\sin\phi + \frac{1}{2}\cos\phi h_{xy}(t,0)\right] \tag{136}$$

By rotating the xy plane around the z axis by $45°$, an $x'y'$ coordinate system is formed as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 45° & \sin 45° \\ -\sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$x'(t) = \frac{1}{\sqrt{2}}r_0(\sin\phi + \cos\phi)\left[1 + \frac{1}{2}h_{xy}(t,0)\right]$$
$$y'(t) = \frac{1}{\sqrt{2}}r_0(\sin\phi - \cos\phi)\left[1 - \frac{1}{2}h_{xy}(t,0)\right] \tag{137}$$

By eliminating the ϕ terms in (137), we derive

$$\frac{x'^2}{[a_\times(t)]^2} + \frac{y'^2}{[b_\times(t)]^2} = 1 \tag{138}$$

which is an ellipse with semi-axes

$$a_\times(t) = \left[1 + \frac{1}{2}h_{xy}(t,0)\right] r_0$$
$$b_\times(t) = \left[1 - \frac{1}{2}h_{xy}(t,0)\right] r_0$$

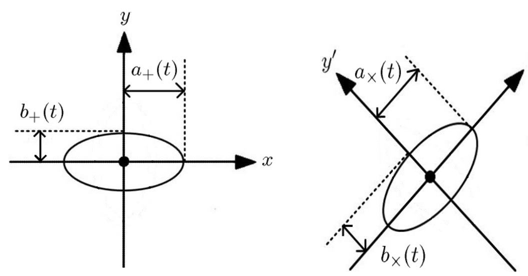

Fig. 5. Elliptical deformations caused by $h_+(t)$ and $h_\times(t)$, respectively.

Thus, both plus polarizations and cross polarizations give rise to elliptical deformation in the distribution of test masses, and the elliptical deformation caused by cross polarization is rotated by $\pi/4$ to that of the plus polarizaton.

A. LIGO

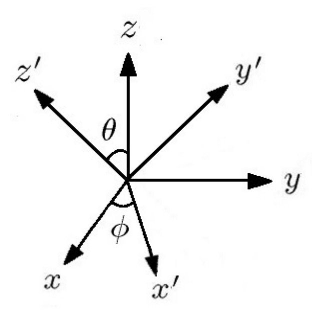

Fig. 6. Coordinates (x', y', z') for incident gravitational wave and (x, y, z) for LIGO instrument.

Fig.6 shows the coordinates (x', y', z') for incident gravitational wave and (x, y, z) for LIGO instrument. The gravitational wave is assumed to propagate along the z' axis, which can be characterized by (θ, ϕ). The coordinate system (x', y', z') can be derived by rotating the coordinate system (x, y, z) by ϕ about the z-axis, and then by θ about the x-axis to have

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (139)$$

Fig.7 shows the configuration of the Advanced LIGO detector

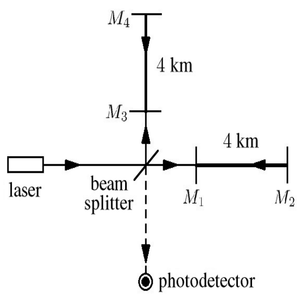

Fig. 7. Configuration of LIGO's interferometer [9].

[9], which is a modified Michelson interferometer designed to measure gravitational wave strain. A coordinate system (ct, x, y, z) is defined, with the two orthogonal arm of the LIGO aligned along the x and y axes, respectively. The separation between mirrors M_1 and M_2 is $\Delta x = L = 4$ km, that between mirrors M_3 and M_4 is $\Delta y = L = 4$ km. The gravitational-wave strain is measured as the difference between $\Delta\ell_x$ and $\Delta\ell_y$ as [10]

$$h(t) = \frac{\Delta\ell_x - \Delta\ell_y}{L} \quad (140)$$

By the definition in (140), we assume a tensor format for the combined response of the two arms of LIGO as [11]

$$h(t) = \frac{1}{2} h_{\mu\nu} A^{\mu\nu} \quad (141)$$

where $h_{\mu\nu}$ is the gravitational wave measured in the LIGO's coordinates, and

$$A^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (142)$$

indicates the components of $h_{\mu\nu}$ measured by the LIGO instrument, namely, $(\Delta\ell_x - \Delta\ell_y)/L$.

Consider a special case that $(x', y', z') = (x, y, z)$, and the gravitational wave is incident in z direction. By substituting (126) into (141), we have

$$h(t) = \frac{1}{2}(h_{xx}A^{xx} + h_{yy}A^{yy}) = h_+(u) \quad (143)$$

Also, by substituting (128) and (129) into (140), we have

$$h(t) = \frac{(1 + h_{xx}/2)\,L - (1 - h_{xx}/2)\,L}{L} = h_+(u)$$

which is the same result as (143).

Let $\bar{u}$ and $\bar{v}$ in the (ct, x, y, z) coordinate system be represented as $\bar{u}'$ and $\bar{v}'$, respectively, in the (ct', x', y', z') coordinate system. If $\bar{u}$ ($\bar{u}'$) is related to $\bar{v}$ ($\bar{v}'$) in the (ct, x, y, z) ((ct', x', y', z')) coordinate system, as

$$\bar{u} = \bar{\bar{h}} \cdot \bar{v} \quad (144)$$

$$\bar{u}' = \bar{\bar{h}}' \cdot \bar{v}' \quad (145)$$

where $\bar{\bar{h}}$ and $\bar{\bar{h}}'$ are the perturbation tensors in the (ct, x, y, z) and the (ct', x', y', z') coordinate systems, respectively.

If the (ct, x, y, z) and the (ct', x', y', z') coordinate systems are related by the rotation in (139), we have

$$\bar{u}' = \bar{\bar{R}} \cdot \bar{u} \quad (146)$$

$$\bar{v}' = \bar{\bar{R}} \cdot \bar{v} \quad (147)$$

where

$$\bar{\bar{R}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ 0 & \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \quad (148)$$

is the rotation matrix derived in (139), with an additional time coordinate. By substituting (144) into (146), and then subtituting (147) into (145), we have

$$\bar{u}' = \bar{\bar{R}} \cdot \bar{u} = \bar{\bar{R}} \cdot \bar{\bar{h}} \cdot \bar{v} = \bar{\bar{h}}' \cdot \bar{\bar{R}} \cdot \bar{v} \quad (149)$$

which implies

$$\bar{\bar{h}} = \bar{\bar{R}}^{-1} \cdot \bar{\bar{h}}' \cdot \bar{\bar{R}} \quad (150)$$

A gravitational wave (GW) with plus polarization propagating in the z' direction is characterized as

$$\bar{\bar{h}}' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(u) & 0 & 0 \\ 0 & 0 & -h_+(u) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (151)$$

which can be transformed into the (ct, x, y, z) coordinate system by substituting (148) and (151) into (150) to have

$$\bar{\bar{h}} = h_+(u) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\phi - \cos^2\theta\sin^2\phi & (1+\cos^2\theta)\sin\phi\cos\phi & \sin\theta\cos\theta\sin\phi \\ 0 & (1+\cos^2\theta)\sin\phi\cos\phi & \sin^2\phi - \cos^2\theta\cos^2\phi & -\sin\theta\cos\theta\cos\phi \\ 0 & \sin\theta\cos\theta\sin\phi & -\sin\theta\cos\theta\cos\phi & -\sin^2\theta \end{bmatrix} \quad (152)$$

Then, the LIGO response to the plus-polarized GW is obtained by substituting (152) into (141) to have

$$h(t) = \frac{1}{2} h_+(u)(1+\cos^2\theta)\cos 2\phi \quad (153)$$

Similarly, a GW with cross polarization propagating in the z' direction is characterized as

$$\bar{\bar{h}}' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_\times(u) & 0 \\ 0 & h_\times(u) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (154)$$

By substituting (148) and (154) into (150), we derive

$$\bar{\bar{h}} = h_\times(u) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\cos\theta\sin 2\phi & \cos\theta\cos 2\phi & \cos\phi\sin\theta \\ 0 & \cos\theta\cos 2\phi & \cos\theta\sin 2\phi & \sin\phi\sin\theta \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & 0 \end{bmatrix} \quad (155)$$

which is substituted into (141) to obtain the LIGO response to a cross-polarized GW as

$$h(t) = -h_\times(u)\cos\theta\sin 2\phi \quad (156)$$

B. Source Localization

The first gravitational wave detection GW150914 was detected by the two LIGO detectors, one in Livingston, Louisiana and the other in Hanford, Washington. For two detectors at different locations, if we assume that the difference in travel time between sites is due only to the direction of the gravitational wave source, then the angle between the source and the detectors baseline can be estimated as

$$\theta = \cos^{-1}\left(\frac{c\tau}{d}\right) \quad (157)$$

where τ is the time delay, and d is the distance between two detectors. The relation in (157) thus constrains the source direction to a ring on the sky. When the arrival time estimates are affected by noise, we can use the Maximum Likelihood Estimator as

$$\hat{\theta} = \begin{cases} \pi, & \tau < -d/c \\ \cos^{-1}(c\tau/d), & -d/c \le \tau \le d/c \\ 0, & \tau > d/c \end{cases} \quad (158)$$

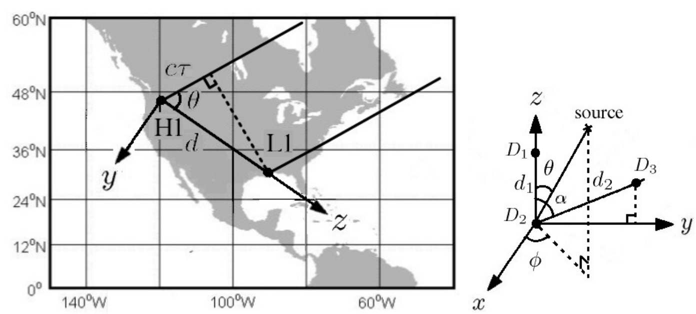

Fig. 8. Illustration of the relation between GW source and detectors baseline in (157), and the coordinate system with positions of three detectors on the $y - z$ plane.

Fig. 8 illustrates the relation between GW source and detectors baseline in (157), and shows a coordinate system with positions of three detectors on the $y - z$ plane.

Assume that the distribution of measured arrival times for a given source follows a gaussain distribution with mean equal to the true arrival time and variance equal to the variance of the arrival time estimate. Define the systematic bias as

$$B_{\hat{\theta}} = <\hat{\theta}> - \theta \quad (159)$$

where $<\hat{\theta}>$ is the expectation value of the estimator in (158), and θ is the true angle between the source and the detectors baseline.

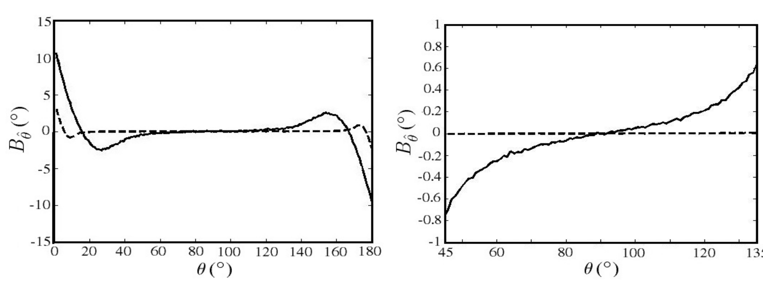

Fig. 9. Angular bias for the Livingston-Hanford (L1-H1) baseline when assuming a gaussian arrival time error distribution with $\sigma = 1$ ms (———) and $\sigma = 0.1$ ms (− − −), respectively.

Fig. 9 shows the angular bias defined in (159) for the Livingston-Hanford (L1-H1) baseline when assuming a gaussian arrival time error distribution with $\sigma = 1$ ms and $\sigma = 0.1$ ms, respectively. We apply twenty thousand sets of arrival times with simulated errors from different sky positions, snd use the distance of the Livingston-Hanford (L1-H1) baseline, where $d/c = 10$ ms.

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