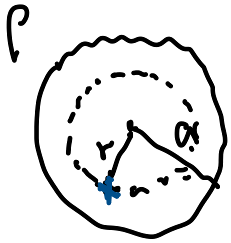
Week 5 Discussion

Wednesday, April 29, 2020 1:57 PM

Uniform density sphere

$$\rho(r) = \begin{cases} \text{const}, & r \le a \\ 0, & r \ge a \end{cases}$$



radial force $F_r(r) = ?$

potential $\phi(r) = ?$

potential energy (PE) = ?

(i) $$F_r = -\frac{GMm}{r^2} = -\frac{Gm}{r^2}\underbrace{\int_0^r \rho \cdot 4\pi r^2\, dr}_{M(<r)}$$

$$= -\frac{Gm}{\cancel{r^2}} \cdot \rho \cdot \frac{4}{3}\pi r^{\cancel{3}}$$

$$= \boxed{-\frac{4\pi G m \rho}{3}\, r}$$

(ii) $$\phi(r) = -\Big[\underbrace{\frac{GM(<r)}{r}}_{\text{inside } r} + \underbrace{4\pi G\int_r^a \rho(r)\cdot r\, dr}_{\text{outside } r}\Big]$$

$$= -\frac{G}{\cancel{r}} \cdot \rho \cdot \frac{4}{3}\pi r^{\cancel{3}\,2} - \cancel{4}^{\,2}\pi G\rho \cdot \frac{1}{\cancel{2}}(a^2 - r^2)$$

$$= -\frac{4}{3}\pi G \rho r^2 - 2\pi G \rho (a^2 - r^2)$$

$$= -2\pi G\rho\left(\frac{2}{3}r^2 + a^2 - r^2\right)$$

$$= \boxed{-2\pi G\rho(a^2 - r^2)}$$

(iii) $PE = \frac{1}{2}\sum_i m_i \phi(r_i) = \frac{1}{2}\int_0^a \phi(r)\cdot\rho\, dV$

$$= \frac{1}{\cancel{2}}\int_0^a -\cancel{2}\pi G\rho^2(a^2 - r^2)\, 4\pi r^2 dr$$

$$= -4\pi^2\rho^2 G \int_0^a \left(a^2r^2 - \frac{r^4}{3}\right) dr$$

$$\frac{a^2}{3}\cdot a^3 - \frac{a^5}{15} = \frac{4}{15}a^5$$

$$= -4\pi^2\rho^2 G\cdot\frac{4}{15}a^5 = -\frac{16}{15}\pi^2 G\rho^2 a^5$$

Substitute $M = \frac{4}{3}\pi a^3\rho$ into PE

$$\Rightarrow \boxed{PE = -\frac{3}{5}\frac{GM^2}{a}}$$

Compare with general form:

$$\boxed{PE = -\frac{GM^2}{2\eta r}} \text{ where } \eta \sim 1$$

$\Rightarrow \frac{1}{2\gamma} = \frac{3}{5} \Rightarrow \gamma = \frac{5}{6} \sim 1$

$\rightarrow$ uniform density sphere.

(iv) obtained PE

$\Rightarrow$ virial theorem to get mass M

$\rightarrow$ check week-4 discussion note.

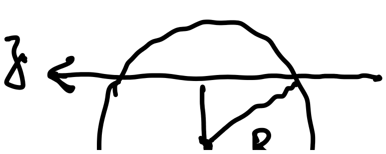
HW4 hints:

- what about Plummer sphere?

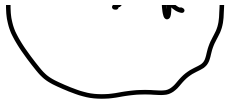
$$\rho_p(r) = \frac{3a_p^2}{4\pi} \cdot \frac{M}{(r^2+a_p^2)^{5/2}}$$

$$\phi_p(r) = -\frac{GM}{\sqrt{r^2+a_p^2}}$$

- surface density $\Sigma(R, S)$



$$= \int_{-\infty}^{\infty} \rho(r) \cdot dz$$



- check week-2 discussion note.

  (surface brightness).