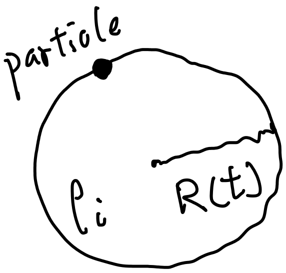
Week 9 Discussion

Wednesday, May 27, 2020 2:00 PM

Expanding Universe

→ uniformly expanding sphere



$$\begin{cases} R(t) = a(t)\cdot r \\ \rho(t) = \rho_i / a^3 \end{cases} \rightarrow \rho_i = \rho(t)\cdot a^3$$

$$M = \frac{4}{3}\pi \rho(t)\cdot R(t)^3 = \frac{4}{3}\pi \underbrace{\rho(t)}\cdot \underbrace{a^3(t)}\cdot r^3 \quad \checkmark$$

$$\text{acceleration } \ddot{R}(t) = -\frac{GM}{R^2} = -\frac{4\pi G \rho_i \cancel{r^3}}{3a^2 \cancel{r^2}}$$

$$= -\frac{4\pi G}{3}\rho_i \cdot \frac{r}{a^2} = \ddot{a}\,\cancel{r}$$

$$\Rightarrow \ddot{a} = -\frac{4\pi G \rho_i}{3a^2} = -\frac{4\pi G \rho(t) a(t)}{3}$$

$\times 2\dot{a}$

$$\Rightarrow 2\dot{a}\ddot{a} = \frac{-8\pi G}{3}\rho(t)\, a(t)\, \dot{a}$$

$$\frac{d}{dt}(\dot{a}^2) \qquad\qquad a^3 \cdot \frac{\dot{a}}{a^2} = -\frac{d}{dt}\left(\frac{1}{a}\right)$$

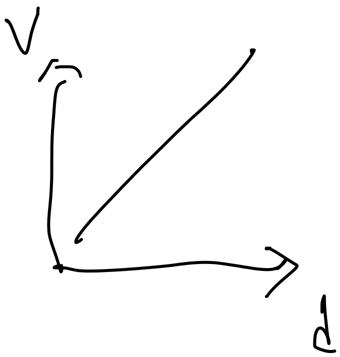
$$\Rightarrow \frac{d}{dt}(\dot{a}^2) = \frac{+8\pi G}{3}\cdot \rho_i \cdot \frac{d}{dt}\left(\frac{1}{a}\right)$$

2

$$\Rightarrow \dot{a}^2 = \frac{8\pi G}{3}\rho_i \cdot \frac{1}{a} = \frac{8\pi G}{3}\rho(t) \cdot a^3 \cdot \frac{1}{\cancel{a}}$$

$$\Rightarrow \boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho(t)} - \underbrace{k}_{=0} c^2/a^2 \quad \longrightarrow \text{flat space.}$$

$$H(t) \equiv \frac{\dot a}{a} \qquad V = H_0 \cdot d$$



$$\int ds = \int a(t) \cdot dr$$

$$\Rightarrow d_p(t) = a(t) \cdot \int_0^r dr = a(t) \cdot r$$

$$\Rightarrow \underbrace{\dot d_p(t)}_{v(t)} = \dot a(t) \cdot r = \underbrace{\boxed{\frac{\dot a}{a}}}_{H(t)} \cdot \underbrace{ar}_{d_p} \checkmark$$

Use 1^{st} law of thermodynamics

$$dQ = dE + PdV = 0$$

→ "adiabetic" due to homogeniety.

$$\frac{dE}{dt} = -P \cdot \frac{da^3}{dt} \quad \text{where } E = \rho a^3 \cdot c^2$$

$$\Rightarrow \frac{d(\rho c^2 a^3)}{dt} = -p \cdot \frac{da^3}{dt}$$

$$\Rightarrow c^2\left(\dot\rho a^3 + \rho \frac{da^3}{dt}\right) = -p \cdot \frac{da^3}{dt}$$

$$\Rightarrow \dot\rho a^3 c^2 = -(p + \rho c^2) \frac{da^3}{dt}$$

$$\Rightarrow \dot\rho = -\frac{(p+\rho c^2)}{a^3 c^2} \cdot 3\cancel{a^2} \cdot \dot a$$

$$= -3\left(\frac{\dot a}{a}\right)\left(\rho + \frac{p}{c^2}\right)$$

pressure

$$\boxed{\dot\rho = -3\left(\frac{\dot a}{a}\right)\left(\rho + \frac{p}{c^2}\right)}$$

(i) Matters : $P_m \sim \rho C_s^2 \Rightarrow \frac{P_m}{c^2} \to 0$

cold, non-relativistic

$$\Rightarrow \dot\rho_m = -3\left(\frac{\dot a}{a}\right)\rho = -3\rho_i a^{-3} \dot a a^{-1}$$

$$= -3 \rho_i a^{-4} \dot{a}$$

$$\rightarrow \rho_m \propto a^{-3}$$

(ii) Photons : $P_\gamma \sim \frac{1}{3} \rho c^2$

$$\Rightarrow \dot{\rho}_\gamma = -\cancel{3}\left(\frac{\dot{a}}{a}\right) \frac{4}{\cancel{3}} \underbrace{\rho_m}_{\rho_i a^{-3}} \underbrace{(1+z)}_{= a^{-1}}$$

$$= -4 \dot{a} \rho_i a^{-5}$$

$$\rightarrow \rho_\gamma \propto a^{-4}$$

(iii) CDM (Λ) : $P_\Lambda \sim -\rho c^2$

$$\Rightarrow \dot{\rho}_\Lambda = 0 \Rightarrow \boxed{\rho_\Lambda = \text{const}}$$

Cosmological models

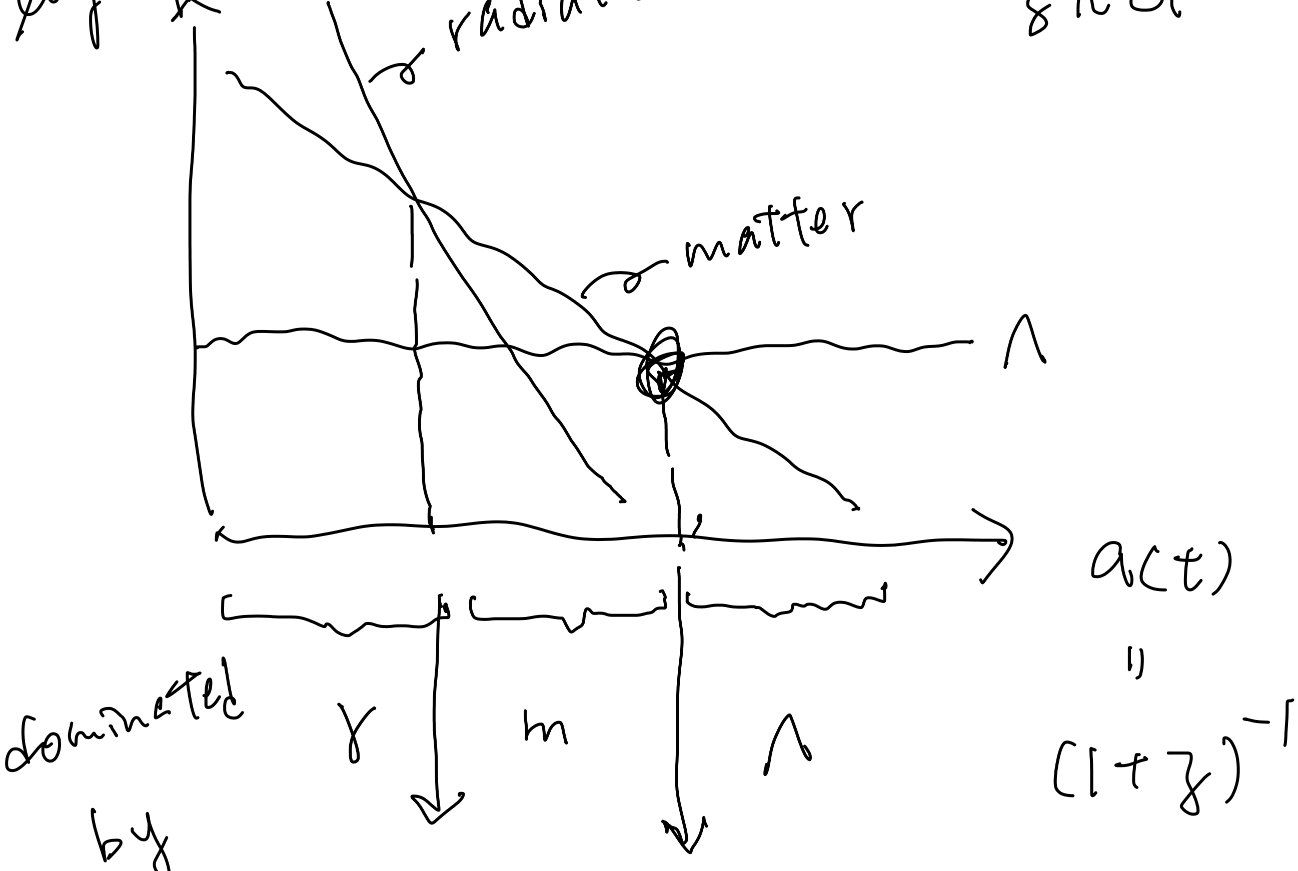
$\Omega_{r0} \sim 8.4 \times 10^{-5}$

$\Omega_{m0} \sim 0.3$

$\Omega_{\Lambda 0} \sim 0.7$

$\underline{H_0} \sim 70 \ \mathrm{km/s \cdot Mpc}$

$\left(\rho_{c0} \equiv \dfrac{3H_0^2}{8\pi G} \right)$



$\underset{\sim}{a_{rm}}$ $\quad$ $\underline{a_{m\Lambda} \simeq 0.773}$

$\downarrow$ $\quad \Rightarrow z_{m\Lambda} \simeq 0.33$

$a_{rm} \simeq 2.8 \times 10^{-4}$

$\rho_\Lambda = \dfrac{\rho_{m0}}{a^3}$

$$\Rightarrow a_{mn}^3 = \frac{\rho_{m0}/\rho_{c0}}{\rho_n/\rho_{c0}}$$

$$\Rightarrow a_{mn} = 0.773$$