Week 4 Problem Session

Thursday, January 28, 2021 10:50 AM

Virial Theorem

virial eq. $2\underbrace{\langle T \rangle}_{\text{mean kinetic energy}} + \underbrace{\langle V \rangle}_{\text{mean potential energy}} = 0$ — $*$

→ mass of a gravitationally bound object

$\boxed{\text{Table 3.1}}$

radial component

M4 cluster: $\begin{cases} \sigma_r = 4 \text{ km/s} \\ r_c = 0.5 \text{ pc} \\ m = 6 \times 10^4 M_\odot \end{cases}$ ✓

$$\langle T \rangle = \frac{1}{2} m \langle \sigma_v^2 \rangle = \frac{3}{2} m \sigma_r^2$$

$$\langle V \rangle = \frac{1}{2} \sum_\alpha m_\alpha \Phi(\alpha) = -\frac{G m^2}{2 r_c}$$

$$* \Rightarrow \cancel{2}\left(\frac{3}{\cancel{2}} \cancel{m} \sigma_r^2\right) = \frac{G m^{\cancel{2}}}{2 r_c}$$

$$\Rightarrow 2\sigma^2 = \frac{Gm}{\ldots} \Rightarrow \boxed{\ldots 6 \sigma_r^2 \cdot r_c}$$

$$\Rightarrow 3\sigma_r^2 = \frac{Gm}{2r_c} \Rightarrow \boxed{m = \frac{6\sigma_r^2 \cdot r_c}{G}}$$

virial mass

$\sigma_r = 4\,\mathrm{km/s}$, $r_c = 0.5\,\mathrm{pc}$

$G = 6.67 \times 10^{-11}$ (SI unit) $m^3\,kg^{-1}\,s^{-1}$

$$m_{vir} = \frac{6 \cdot \underbrace{(4000)}_{}{}^{\overset{m/s}{}\,2}\;\overset{pc \rightarrow m}{\underbrace{(0.5 \times 3.086 \times 10^{16})}}}{6.67 \times 10^{-11}}$$

$$= 2.22 \times 10^{34} \quad (kg)$$

$$= \frac{2.22 \times 10^{34}\,(kg)}{2 \times 10^{30}\quad(kg)} \sim \underline{\underline{10^4\,M_\odot}}$$

$\updownarrow$

$m \doteqdot \underline{\underline{6 \times 10^4\,M_\odot}}$

$$\boxed{M = \frac{6\sigma_r^2\,\eta\,r_c}{G}} \rightarrow \eta \sim 1$$

depend on density distributions

Conversion between $\underbrace{\rho}_{\text{density}}$ and $\underbrace{\Phi}_{\text{gravitational potential}}$

HW: given $\Phi \rightarrow$ obtain ρ

Poisson's eq: $\nabla^2 \Phi = 4\pi G \rho$

$$\Rightarrow \rho = \frac{\boxed{\nabla^2 \Phi}}{4\pi G}$$

$\leftarrow$ coordinate (spherical, cylindrical, Cartesian)

given $\rho \rightarrow$ obtain Φ ?

"singular isothermal sphere"

spherical density distribution

$$\rho_{SIS} = \frac{\boxed{\rho(r_0)}}{(r/r_0)^2}, \quad \rho(r_0) = \frac{V_H^2}{4\pi G r_0^2}$$

r_0 is constant (HW3 problem 1 $a_H \rightarrow 0$)

show that $\Phi_{SIS} = \underline{V_H}^2 \ln\left(\frac{r}{r_0}\right)$

$$\phi(r) \sim -\frac{GM}{r} = \underbrace{-\int \frac{G\rho(x')}{x - x'} d^3x'}$$

$$\Rightarrow -\int \frac{G\rho_{sis}}{\cancel{r}} \underbrace{dV}_{4\pi r^2 dr}$$

$$\rho(r) = \frac{V_H^2}{4\pi G r^2}$$

$$= -\int G\rho_{sis} \cdot 4\pi r\, dr$$

$$\rho_{sis} = \frac{\boxed{\rho(r_0)}}{(r/r_0)^2}$$

$$= -\int_r^{r_0} 4\pi G \rho_{sis}\, r\, dr$$

$$= -\underbrace{4\pi G r_0^2 \rho(r_0)}_{V_H^2} \int_r^{r_0} \frac{\cancel{r}}{r^{\cancel{2}}} dr$$

$$= -V_H^2 \cdot \ln(r)\Big|_r^{r_0} = V_H^2 \ln\left(\frac{r}{r_0}\right)$$

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