Week 5 Problem Session

Thursday, February 4, 2021 10:52 AM

last week: virial theorem

relation $\Phi \longleftrightarrow \rho$

$\Rightarrow \nabla^2 \Phi = 4\pi G \rho$

$\Updownarrow$

$$\Phi(x) = -\int \frac{G\rho(x')}{|x - x'|} d^3x'$$

If we have $\rho(r)$

$\rightarrow$ gravitational force (F)

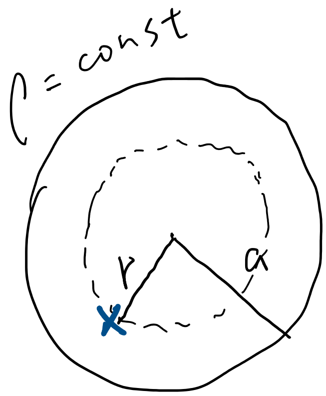
$\rightarrow F = -\nabla \Phi \Rightarrow \Phi$: grav. potential

$\rightarrow \Phi = \frac{1}{2} \sum_i m_i \Phi(r_i)$ grav. potential energy (PE)

uniform density sphere

virial theorem

$$\rho(r) = \begin{cases} \rho = \text{const}, & r \le a \\ 0, & r \ge a \end{cases}$$



(i) radial force $F_r(r) = ?$

(ii) potential $\Phi(r)$

(iii) PE = ?

(iv) virial thm. $2\langle KE \rangle + \langle PE \rangle = 0$

(i) $$F_r = -\frac{GMm}{r^2} = -\frac{Gm}{r^2}\underbrace{\int_0^r \rho \cdot 4\pi r^2 \, dr}_{M(<r)}$$

$$= -\frac{Gm}{\cancel{r^2}}\rho \cdot \frac{4}{3}\pi r^{\cancel{3}}$$

$$= \boxed{-\frac{4\pi G m \rho}{3} r}$$

(ii) $$\Phi(r) = -\left[\underbrace{\frac{GM(<r)}{r}}_{\text{inside } r} + \underbrace{4\pi G \int_r^a \rho(r) \cdot r \, dr}_{\text{outside } r}\right]$$

$$= -\frac{G}{\cancel{r}} \cdot \rho \cdot \frac{4}{3}\pi r^{\cancel{3}\,2} - \cancel{4}^{2}\pi G \rho \cdot \frac{1}{\cancel{2}}(a^2 - r^2)$$

$$= -\frac{4}{3}\pi G \rho r^2 - 2\pi G \rho (a^2 - r^2)$$

$$= -2\pi G \rho \left(\frac{2}{3}r^2 + a^2 - r^2\right)$$

$$= \boxed{-2\pi G \rho (a^2 - r^2/3)}$$

(iii) $PE = \frac{1}{2}\sum_i m_i \phi(r_i) = \frac{1}{2}\int_0^a \phi(r) \cdot \rho \, dV$

$$= \frac{1}{\cancel{2}}\int_0^a -\cancel{2}\pi G \rho \, (a^2 - r^2/3) \cdot \rho \cdot 4\pi r^2 dr$$

$$= -4\pi^2 \rho^2 G \int_0^a \underbrace{(a^2 - r^2/3) r^2}_{a^2 r^2 - r^4/3} dr$$

$$= -4\pi^2 \rho^2 G \cdot \left[\frac{a^2}{3}(a^3) - \frac{a^5}{15}\right]$$

$a^5 \quad a^5 \quad 4 \quad a^5$

$$\frac{a^5}{3} - \frac{a^5}{15} = \frac{4}{15} a^5$$

$$= \underbrace{-\frac{16}{15} \pi^2 G \rho^2 a^5}$$

Substitute $M = \frac{4}{3} \pi a^3 \rho$ into P.E.

$$\boxed{PE = -\frac{3}{5} \frac{GM^2}{a}}$$

Compare with general form:

$$\boxed{PE = -\frac{GM^2}{2\eta r}}, \quad \text{where } \eta \sim 1$$

For uniform density sphere,

$$\frac{1}{2\eta} = \frac{3}{5} \Rightarrow \boxed{\eta = \frac{5}{6}} \sim 1$$

(iv) obtained PE $\Rightarrow$ virial theorem to get mass M.

$$\cancel{2} \cdot \frac{\cancel{3} \cancel{M} \sigma_r^2}{\cancel{2}} = \frac{\cancel{3}}{5} \frac{G M^{\cancel{2}}}{a}$$

$$\Rightarrow \boxed{M_{vir} = \frac{5 a \sigma_r^2}{G}}$$

HW4 hints

- what about Plummer sphere?

$$\rho_p(r) = \frac{3 a_p^2}{4\pi} \cdot \frac{M}{(r^2 + a_p^2)^{5/2}}$$

$$\Phi_p(r) = -\frac{GM}{\sqrt{r^2 + a_p^2}}$$

- surface brightness

mag/ $\longleftrightarrow$ $L_\odot$/

$\mathrm{mag}/\mathrm{arcsec}^2 \longleftrightarrow L_\odot/\mathrm{pc}^2$

$\Rightarrow$ check notes of week 2.

- surface density $\Sigma(R, S)$

$$= \int_{-\infty}^{\infty} \rho(r) \cdot dz.$$