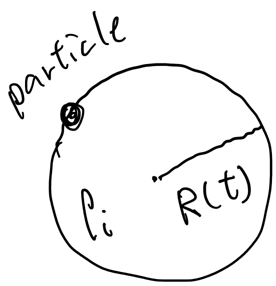
Week 9 Problem Session

Thursday, March 4, 2021 10:55 AM

Expanding Universe

→ uniformly expanding sphere



$$\begin{cases} R(t) = a(t)\cdot r \\ \rho(t) = \rho_i / a^3 \end{cases} \rightarrow \rho_i = \rho(t)\cdot a^3$$

$$M = \frac{4}{3}\pi \rho(t)\cdot R^3(t) = \frac{4}{3}\pi \rho(t)\, a^3(t)\cdot r^3$$

$$\text{acceleration } \ddot{R}(t) = -\frac{GM}{R^2} = -\frac{4\pi G \rho_i}{3a^2\cdot \cancel{r^2}} r^3$$

$$= -\frac{4\pi G}{3}\rho_i \cdot \frac{\cancel{r}}{a^2} = \ddot{a}\,\cancel{r}$$

$$\Rightarrow \ddot{a} = -\frac{4\pi G \rho_i}{3a^2} = -\frac{4\pi G \rho(t)\, a(t)}{3}$$

$\times 2\dot{a}$

$$\Rightarrow 2\dot{a}\ddot{a} = -\frac{8\pi G}{3}\rho(t)\, a(t)\, \dot{a}$$

$$\frac{d}{dt}(\dot{a}^2) \qquad\qquad a^3 \cdot \boxed{\frac{\dot{a}}{a^2}} = -\frac{d}{dt}\left(\frac{1}{a}\right)$$

$$\Rightarrow \frac{d}{dt}(\dot{a}^2) = \frac{8\pi G}{3}\underbrace{\rho(t)\, a^3}_{\rho_i}\frac{d}{dt}\left(\frac{1}{a}\right)$$

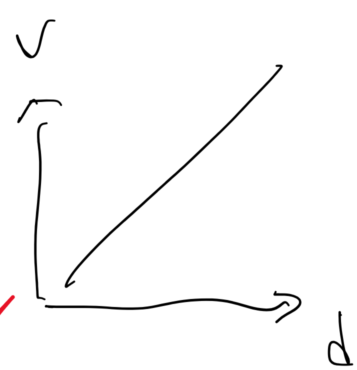
$$\Rightarrow \dot{a}^2 = \frac{8\pi G}{3}\rho(t)\, a^{\cancel{3}\,2}\frac{1}{\cancel{a}} = \frac{8\pi G}{3}\rho(t)\cdot a^2$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t)} \rightarrow \text{flat spacetime}$$

$$- k\, c^2/a^2 \qquad k = 0$$

Friedmann Equation

$$H(t) \equiv \frac{\dot{a}}{a} \qquad\qquad V = H_0 \cdot d \quad ✓$$



$$\int ds = \int a(t)\cdot dr$$

$$\Rightarrow d_p(t) = a(t)\int_0^r dr = a(t)\cdot r$$

$$\Rightarrow \underbrace{\dot{d}_p(t)}_{v(t)} = \dot{a}(t)\, r = \underbrace{\boxed{\frac{\dot{a}}{a}}}_{=}\;\underbrace{a r}_{d_p} \quad ✓$$

$v(t)$ $\qquad$ $\underbrace{\quad}_{= H(t)}$ dp

// Einstein Field Equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \underline{T_{\mu\nu}}$$

stress-energy tensor

$$\Gamma^{\alpha}_{\beta\gamma} \quad \Gamma^{\alpha}_{\beta\gamma,\delta} \ni g_{\mu\nu} \text{ or } g_{\mu\nu,\sigma}$$

flat : $ds^2 = -dt^2 + a^2(t) \cdot dx^2$

Continuity equation

In GR, $\nabla_\mu T^{\mu\nu} = 0$

Use 1^{st} law of thermodynamics

$$dQ = dE + PdV = 0$$

$\rightarrow$ "adiabatic" due to homogeniety.

$$\frac{dE}{dt} = -p\,\frac{da^3}{dt} \quad \text{where } E = \rho a^3 \cdot c^2$$

$$\Rightarrow \frac{d(\rho c^2 a^3)}{dt} = -p\,\frac{da^3}{dt}$$

$$\Rightarrow c^2\left(\dot{\rho} a^3 + \rho \frac{da^3}{dt}\right) = -p\,\frac{da^3}{dt}$$

$$\Rightarrow \dot{\rho} a^3 c^2 = -(p + \rho c^2)\,\frac{da^3}{dt}$$

$$\Rightarrow \dot{\rho} = -\frac{p + \rho c^2}{a^{\cancel{3}} c^2}\, 3\cancel{a^2} \cdot \dot{a}$$

$$= -3\left(\frac{\dot{a}}{a}\right)\left(\rho + \frac{p}{c^2}\right) \rightarrow \text{continuity equation}$$

☆ $$\boxed{\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)\left(\rho + \frac{p}{c^2}\right)}$$ pressure

$$p = w \cdot \rho c^2$$

$$\Rightarrow \dot{\rho} + 3H(1+w)\rho = 0$$

$$\Rightarrow \rho(t) = \rho_0\, a^{-3(1+w)}$$

(i) Matters (cold, non-relativistic)

$$P_m \sim \rho \underbrace{c_s^2}_{\ll c} \Rightarrow \frac{P_m}{c^2} \to 0 \Rightarrow w = 0$$

$$\Rightarrow \rho_m = \rho_0 a^{-3}$$

(ii) Photons: $P_\gamma \sim \frac{1}{3} \rho c^2 \Rightarrow w = \frac{1}{3}$

$$\Rightarrow \rho_\gamma = \rho_0 a^{-4}$$

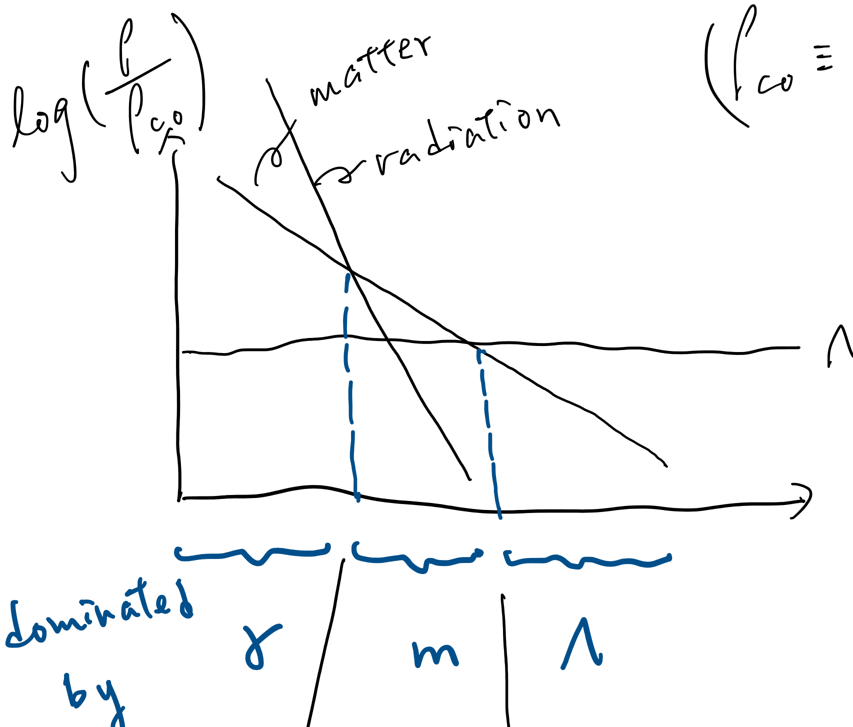
(iii) vacuum energy: $P_\Lambda \sim -\rho c^2 \Rightarrow w = -1$

$$\Rightarrow \rho_\Lambda = \rho_0 = \text{const}$$

Standard cosmology

$$\left.\begin{aligned} &\Omega_{m0} \sim 0.3 \\ &\Omega_{\Lambda 0} \sim 0.7 \\ &\Omega_{r0} \sim 8 \times 10^{-4} \\ &H_0 \sim 70 \ \mathrm{km/s \cdot Mpc} \end{aligned}\right\}$$

$$\left(\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}\right)$$



$a(t) = (1+z)^{-1}$

a_{rm}

$\downarrow$

$a = 2.8 \times 10^{-4}$

$\Rightarrow z \sim 3500.$

$a_{m\Lambda} = 0.77$

$\Rightarrow z_{m\Lambda} = 0.3$

$$\rho_\Lambda = \frac{\rho_{m0}}{a_{m\Lambda}^3} \Rightarrow a_{m\Lambda}^3 = \frac{\rho_{m0}/\rho_{c0}}{\rho_\Lambda/\rho_{c0}}$$

$\Rightarrow \zeta \sim$ ... m

$\Rightarrow$ Am...

$$\int_{t_e}^{t_0} dt = \int_0^{\zeta} \boxed{} \, d\zeta$$

$t_e = \zeta$